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Uncertainty and Complementarity in Axiomatic Quantum Mechanics

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In this work an investigation of the uncertainty principle and the complementarity principle is carried through. A study of the physical content of these principles and their representation in the conventional Hilbert space formulation of quantum mechanics forms a natural starting point for this analysis. Thereafter is presented more general axiomatic framework for quantum mechanics, namely, a probability function formulation of the theory. In this general framework two extra axioms are stated, reflecting the ideas of the uncertainty principle and the complementarity principle, respectively. The quantal features of these axioms are explicated. The sufficiency of the state system guarantees that the observables satisfying the uncertainty principle are unbounded and noncompatible. The complementarity principle implies a non-Boolean proposition structure for the theory. Moreover, nonconstant complementary observables are always noncompatible. The uncertainty principle and the complementarity principle, as formulated in this work, are mutually independent. Some order is thus brought into the confused discussion about the interrelations of these two important principles. A comparison of the present formulations of the uncertainty principle and the complementarity principle with the Jauch formulation of the superposition principle is also given. The mutual independence of the three fundamental principles of the quantum theory is hereby revealed.

Energia, massa;
aika loputon...
Onko sielu olemassa?
Mikä maailman muoto on?

Osmo Lahti

1. INTRODUCTION

In the foundations of quantum theory there exist three fundamental principles: the superposition principle, the uncertainty principle, and the complementarity principle.

The superposition principle, which is characteristic for any wave theory, was introduced into quantum theory through the works of Louis de Broglie [matter–wave hypothesis of 1924; de Broglie, 1925] and Erwin Schrödinger (1926a, b). The full appreciation of the quantum mechanical superposition principle is, however, due to Paul Dirac. Indeed, in his book *The Principles of Quantum Mechanics* (originally published in 1930) Dirac shows that the principle of superposition of states is one of the most fundamental properties of quantum mechanics. In this work he also underlines the crucial difference between the superpositions in classical theory and in quantum theory. The mathematical structure that Dirac gave to the superposition principle was linearity:

> ...each state of a dynamical system at a particular time corresponds to a ket vector, the correspondence being such that if a state results from the superposition of certain other states, its corresponding ket vector is expressible linearly in terms of the corresponding ket vectors of the other states, and conversely (Dirac, p. 16).[1]

Dirac's approach to quantum mechanics was later adopted by many authors, among them Albert Messiah (1961) and Bernard d'Espagnat (1976).

The superposition principle has also been studied in the quantum logic approach to axiomatic quantum mechanics (see, e.g., Jauch 1968; Varadarajan, 1968; Gudder, 1970a; and Beltrametti and Cassinelli, 1976). In these investigations the importance of the superposition principle in quantal description has further been clarified. These investigations have revealed among other things that the quantum mechanical superposition principle implies the non-Boolean character of the proposition system (see, e.g., Jauch, 1968, p. 107). Moreover, the crucial difference between the superpositions in classical theory and in quantum theory is now completely appreciated; in classical mechanics every superposition is a mixture, whereas in quantum mechanics the existence of superpositions of states which are not mixtures is relevant (see, e.g., Beltrametti and Cassinelli, 1976, pp. 377–381).

The uncertainty principle and the complementarity principle originated through the works of Werner Heisenberg (1927) and Niels Bohr (1927). With these two principles two solutions to the interpretation problem of the quantum theory were provided.

According to Heisenberg a consistent application of the concepts of classical physics in the quantum domain was secured by posing some

[1]We refer to the 1958 reprint of Dirac's work originally published in 1930.

limitations to the simultaneous measurability of certain physical quantities. These limitations Heisenberg expressed in his famous uncertainty relations.

According to Bohr the wave–particle duality was so central a phenomenon that it should be the natural starting point for any interpretation of the quantum theory (cf. the origin of the superposition principle). Starting from this duality Bohr developed his notion of complementarity, which was to "denote the relation of mutual exclusion characteristic of the quantum theory with regard to the application of the various classical concepts and ideas" (Bohr, 1929/1978, p. 19).

In the Hilbert space formulation of quantum mechanics the uncertainty principle is expressed in the inequality

$$\mathrm{Var}(A,\phi)\mathrm{Var}(B,\phi) \geqslant \tfrac{1}{4}|(\phi,(AB-BA)\phi)|^2,$$
$$\forall\phi\in\mathrm{dom}(AB)\cap\mathrm{dom}(BA) \qquad (1.1)$$

whereas the complementarity principle is quite generally regarded as "an extraneous interpretative addition to it" (Jammer, 1974, p. 60).

In quantum logics the uncertainty principle and the complementarity principle have not met with such a penetrating investigation as the superposition principle. It is true that the uncertainty principle and the complementarity principle have been used in quantum logics, too, e.g., in discussing the choice of the syntactic rules for the proposition system, but a detailed analysis of these two principles has not thus far been carried out. This will be the aim of this work.

In Sections 2 and 3 of this work the uncertainty principle and the complementarity principle are discussed. It is our aim to give to these two principles formulations which would quite easily lead to mathematical expressions in a certain axiomatic framework for quantum mechanics. When such formulations are given one is in position to carry out an active study of these principles in the given framework. Also the comparison of the two principles can then be made on solid ground.

Independently of some interpretative questions, nonstatistical interpretation versus statistical interpretation, the uncertainty principle does not give rise to any particular difficulty in this respect. The uncertainty relations, the above-mentioned inequality (1.1) provide the mathematical formulation of the uncertainty principle.

The complementarity principle is, however, much more problematic. First of all, there is no generally accepted formulation of the complementarity principle. Though Bohr introduced the notion of complementarity in 1927 to acquire a consistent interpretation of the quantum theory, and though he published in the following 35 years a series of essays in which he strove to develop the idea of complementarity into a definite philosophical viewpoint, he never gave an explicit definition of his notion of complementarity. Thus our first task is to form a compact picture of the viewpoint of complementarity. After analyzing Bohr's writings on complementarity we conclude that the relation of mutual exclusion is the most important element in the notion of complementarity. Based on this relation we give a definition of complementary physical quantities. The complementarity principle then expresses the view that the existence of such quantities is essential in the quantal description of any physical system.

In Section 4 we discuss the two principles in the conventional Hilbert space formulation of quantum mechanics. It is well known that for any two self-adjoint operators A and B acting on a Hilbert space H the inequality (1.1) holds, which is the uncertainty relation for the corresponding physical quantities. The spectral measures P^Q and P^P of the canonically conjugate position and momentum operators Q and P are shown to have the property

$$P^Q(E)\wedge P^P(F)=0 \quad \text{for every bounded } E \text{ and } F \text{ in } B(R) \qquad (1.2)$$

which indicates the complementary nature of these quantities. Thus in the Hilbert space formulation of quantum mechanics the complementary nature of two physical quantities appears to be a derivable property, too. This result seems to be of some interest because it is quite generally taught that the complementarity principle is an extraneous interpretative addition to the Hilbert space formulation of quantum mechanics, and that the uncertainty relations exhibit a mathematical expression for the complementarity principle.

After giving a short sketch in Section 5 of the probability function formulation of axiomatic quantum mechanics, we discuss in Sections 6 and 7, respectively, the uncertainty principle and the complementarity principle in that general framework. We shall state the two principles as the axioms of the theory. It will then be shown that both the axiom of Heisenberg and the axiom of complementarity, which exhibit the ideas of uncertainty and complementarity, respectively, are both of a quantal nature, thus excluding the classical mechanical description of any physical system. The sufficiency of the state system guarantees that the observables satisfying the uncertainty principle are unbounded, and noncompatible. The complementarity principle implies a non-Boolean proposition structure for the theory. Moreover, nonconstant complementary observables are always noncompatible.

The logical independence of the axiom of Heisenberg and the axiom of complementarity is shown in Section 8. Thus some order is brought to the confused discussion about the interrelations of these two important principles. In actual fact, in the relevant literature the view that the complementarity principle contains the uncertainty principle is frequently put forward, but also the converse view has its advocates.

In the final Section of this work we discuss the "logical status" of the two axioms. Starting from the classical Hamiltonian description of a given physical system we pass on to its quantum mechanical description. It is argued that in this "transition" the effect of the uncertainty principle is to deform the concept of state, whereas the effect of the complementarity principle is to deform the propositional structure. This seems to be satisfactory because the uncertainty principle is of a statistical character, and the complementarity principle is of a nonstatistical character. Thus we are also quite naturally led to distinguish between the "uncertainty description" and the "complementary description," both of which embrace an important feature of the quantal description, but which only together provide the proper quantum mechanical description of any physical system. Using Jauch's formulation for the superposition principle we shall also show in Section 9 that the superposition principle is logically independent from the uncertainty principle and the complementarity principle as formulated in this work.

2. ON THE UNCERTAINTY PRINCIPLE

The clarification of the conceptual foundations of quantum theory began with the publication of Werner Heisenberg's historic paper "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik" in 1927. With this paper Heisenberg provided an intuitive understanding of the fundamental quantum mechanical relations, especially that of the "exchange relation" $pq-qp=h/2\pi i$. The key to this intuitive understanding—as Heisenberg has more recently revealed (Heisenberg, 1955, 1958, 1967, 1977) was in his recognition that only such experimental situations can arise in nature which can be expressed in the mathematical formalism of quantum mechanics.

2.1. Heisenberg's 1927 Paper. In the abstract preceding the paper (Heisenberg, 1927) Heisenberg promises to give exact definitions of the words "position," "velocity," "energy," etc. (e.g., of an electron), which are valid also in quantum mechanics, and to show that the canonically conjugate quantities can simultaneously be determined only with a characteristic uncertainty. According to Heisenberg this uncertainty, which he showed to be a mathematical consequence of the Dirac–Jordan transformation theory, is the essential reason for the occurrence of statistical relations in quantum mechanics.

The need for a reinterpretation of the kinematical and mechanical concepts in the quantum domain was evident for Heisenberg from the fundamental commutation relation $pq-qp=h/2\pi i$. In analyzing these concepts Heisenberg adopted the operational (or instrumentalist) view of reducing the definability of a physical concept to its measurability. This can be read from the following citation of Heisenberg:

> If one wants to clarify what is meant by "position of an object," e.g., of an electron (relative to a given reference system), one has to describe an experiment by which the "position of an electron" can be measured; otherwise this concept has no meaning at all (Heisenberg, 1927, p. 174).

We are not, as Heisenberg pointed out, lacking experiments which allow us to define the concept "position of an electron." For example, the position of an electron can be measured with any desired degree of accuracy by means of a microscope employing radiation of sufficiently short wavelength, a gamma-ray microscope promising optimal accuracy.

The Compton effect, however, plays an essential role in such measurements, and Heisenberg writes:

> At the moment of the position determination, that is, when the light quantum is scattered by the electron, the electron changes its momentum discontinuously. This change is the greater, the smaller is the wave length of the light used, i.e. the more accurate is the position determination. Thus, at the moment when the position of the electron is known, its momentum can be known only up to a magnitude corresponding to the discontinuous change; so, the more accurate is the position determination, the less accurate is the momentum determination and vice versa (Heisenberg, 1927, p. 175).

Denoting by q_1 and p_1 the accuracies in the position determination (the wavelength of the illuminating light) and in the momentum determination (the discontinuous change in the momentum due to the Compton effect), respectively, Heisenberg ended with his famous relation $q_1 p_1 \sim h$.

Heisenberg analyzed also a Stern–Gerlach experiment for the determination of the magnetic moment of an atom to show that the uncertainty E_1 in the energy measurement is the greater, the shorter is the time t_1 spent by the atom in crossing the deviating field: in symbols, $E_1 t_1 \sim h$. This relation indicates "how an accurate energy determination can be obtained only by a corresponding uncertainty in time" (Heisenberg, 1927, p. 179).

From the above considerations Heisenberg drew the following conclusion:

> All the concepts that are used in the classical theory for the description of a mechanical system can also be defined exactly for atomic processes. But the experiments which allow such definitions carry with them an uncertainty if they involve the simultaneous determination of two canonically conjugate quantities (Heisenberg, 1927, p. 179).

The next step in Heisenberg's reasoning was to show that the relation $q_1 p_1 \sim h$, which acquired "a direct intuitive interpretation for the relation $pq-qp=h/2\pi i$" (Heisenberg, 1927, p. 175), is also derivable from the quantum theoretical formalism. To do this Heisenberg applied the Dirac–Jordan transformation theory. Assuming now a Gaussian distribution for the position coordinate q of the electron, he calculated, following Jordan's method, the corresponding distribution for the momentum p of the electron. This distribution turned out to be a Gaussian distribution, too. Identifying now the uncertainties q_1 and p_1 in position and in momentum with the half-widths of the Gaussian curves Heisenberg found that $q_1 p_1 = h/2\pi$.

In the above-described way Heisenberg established that "the relation $q_1 p_1 \sim h$ is in direct mathematical connection with the exchange relation $pq-qp=h/2\pi i$" (Heisenberg, 1927, p. 175), and that it thus offered its "direct intuitive interpretation" (Heisenberg, 1927, p. 175).

2.2. Bohr versus Heisenberg. In determining the position of an electron with the gamma-ray microscope Heisenberg referred to the discontinuous change of momentum due to the Compton effect as a reason for the uncertainty in the momentum of the electron. However, as Bohr pointed out and Heisenberg admitted in the postscript to his paper, the uncertainty in momentum is essentially due to "the necessary divergence of the radiation beam" (Heisenberg, 1927, p. 198) under the microscope.

In addition to this apparent deficiency in Heisenberg's treatment of his thought experiment this paper gave rise to a fundamental disagreement between Heisenberg and Bohr.[2] Though Bohr did not question the validity of the uncertainty relations he did not agree with Heisenberg about the conceptual foundations on which they were founded.

[2] Heisenberg has given in his more recent article (Heisenberg, 1967) a very readable account of the discussions he had with Bohr on the meaning of the uncertainty relations. The two books of Jammer (1966, 1974) also contain much valuable information about this subject matter.

The origin of this disagreement can be seen to reside in the fact that Bohr had progressed in a somewhat different direction than Heisenberg in looking for the solution to the interpretation problem of the quantum theory. We shall discuss Bohr's approach in more detail in the next section.

Bohr's point of departure was the fundamental wave–particle duality, which he regarded as being so central a phenomenon that it should be the natural starting point for any interpretation of the quantum theory.

Bohr argued that every derivation of the uncertainty relations from the analysis of thought experiments must somewhere have recourse to the Einstein–de Broglie equations, for otherwise the whole reasoning would remain classical and no uncertainty relations could be obtained. The Einstein–de Broglie relations $E=h\nu$, $p=h/\lambda$ connect in a striking way particle attributes with wave attributes and thus express the wave–particle dualism.

Thus in Bohr's view the uncertainty relations indicate that the wave–particle duality is the ultimate foundation of the whole theory. On the other hand, Heisenberg, knowing that the uncertainty relations are logically derivable from the mathematical formalism, did not consider the wave-particle duality as a necessary presupposition of the theory.

The other, though certainly related, disagreement between Bohr and Heisenberg is usually characterized by stating that for Heisenberg the uncertainty relations indicated limitations of measurement and for Bohr they indicated limitations of definition (see, e.g., Petersen, 1968, p. 110).

Heisenberg's point of departure was, as already discussed, his reinterpretation of classical concepts in the quantum domain by reducing the definability of a physical concept to its measurability. Heisenberg then ended with the conclusion that "all the concepts that are used in the classical theory for the description of a mechanical system can also be defined [i.e., measured] exactly for atomic processes. ...But," Heisenberg continues, "the experiments which allow such definitions [i.e., measurements] carry with them an uncertainty if they involve the simultaneous determination of two canonically conjugate quantities" (Heisenberg, 1927, p. 179).

Bohr's point of departure was, as already referred to, the fundamental wave–particle duality. This duality, which is necessarily involved in every derivation of the uncertainty relations from thought experiments, gives us the limits within which we can without contradiction use classical concepts in describing quantum phenomena. In this connection, when speaking about conjugate quantities, Bohr declared that "the reciprocal uncertainty which always affects the values of these quantities is essentially an outcome of the limited accuracy with which changes in energy and momentum can be defined" (Bohr, 1927/1978, p. 63).

After several weeks of painful discussion, as Heisenberg has afterwards revealed (Heisenberg, 1967, p. 106), agreement was, however, reached. Heisenberg agreed with Bohr that "the uncertainty relations were just a special case of the more general complementarity principle" (Heisenberg, 1967, p. 106), which made it possible to take wave–particle dualism as a suitable starting point for an interpretation of the quantum theory.

2.3. Popper versus Heisenberg. The uncertainty relation $q_1 p_1 \sim h$, which Heisenberg derived through his gamma-ray microscope thought experiment, refers to a single measurement made on a single physical system. On the other hand, the mathematical derivation of the uncertainty relation $q_1 p_1 = h/2\pi$ through the Dirac–Jordan transformation theory is based essentially on Born's probability interpretation of the Schrödinger wave function. Hence the latter relation $q_1 p_1 = h/2\pi$ is, contrary to the former, of a statistical character. Thus the connection between the two derived uncertainty relations $q_1 p_1 \sim h$ and $q_1 p_1 = h/2\pi$ is not so obvious.

In fact, as Jammer has emphasized, Heisenberg's identification of the statistical formula $q_1 p_1 = h/2\pi$ with the relation $q_1 p_1 \sim h$ did not have the character of a logically necessary conclusion (Jammer, 1966, p. 330).

We have now learned that Heisenberg interpreted the uncertainty relations as penetrating to individual particles and not as a statistical spread of the results obtained when measuring the positions and momenta of the members of an ensemble of particles.

On the other hand, Heisenberg had argued that these relations "can also be deduced...from the mathematical scheme of quantum theory and its physical interpretation" (Heisenberg, 1930/1949, p. 15). The physical interpretation which Heisenberg had in mind in this connection was clearly stated in his Chicago lectures in connection with his discussion of the concept of an orbit of an electron in an atom. Heisenberg wrote:

> One can repeat this single observation on a large number of atoms, and thus obtain a probability distribution of the electron in the atom. According to Born, this is given mathematically by $\psi^*\psi$. This is the physical significance of the statement that $\psi^*\psi$ is the probability of observing the electron at a given point (Heisenberg, 1930/1949, p. 33).

So it would have been very natural, or rather logical, for Heisenberg to have concluded that the physical significance of the uncertainty relations is of a similar character to that of $\psi^*\psi$, i.e., they express a statistical scatter relation. However, as we know, Heisenberg did not do that. He preferred to "make uncertainty the central point in the interpretation"

(Heisenberg, 1967, p. 105) of the quantal formalism, and to regard this uncertainty as "the essential reason for the occurrence of the statistical relations in quantum mechanics" (Heisenberg, 1927, p. 172).

The above view was firmly opposed by Karl Popper. Popper thought that an analysis, like that of Heisenberg's, of the relations between the uncertainty formulas and the statistical interpretation of the quantum theory was not acceptable. In 1934 he wrote:

> It seems to me that the logical relation is just the other way round. For we can derive the uncertainty formulae from Schrödinger's wave equation (which is to be interpreted statistically), but not this latter from the uncertainty formulae. If we are to take due account of these relations of derivability, then the interpretation of the uncertainty formulae will have to be revised. ...It is true that Heisenberg's formulae result as logical conclusions from the theory, but the interpretation of these formulae as rules limiting attainable precision of measurement, in Heisenberg's sense, does not follow from the theory (Popper, 1968, pp. 223 and 224).

In this connection Popper suggested a statistical reinterpretation of Heisenberg's formulas according to which they express merely statistical scatter relations between the parameters involved (see Popper, 1968, pp. 223–225).

2.4. Further Developments. The great importance of Heisenberg's uncertainty relations in physics was realized very soon. The uncertainty relations, or the *uncertainty principle* as they also were called, became a much-discussed subject in physics and subsequently also in philosophy. Soon after Heisenberg's fundamental work in 1927 more general derivations of these relations were also available. Moreover, the discussion of their physical interpretation (see, e.g., Popper's interpretation above) as well as their philosophical implications was very lively. So today we really have an extensive literature dealing with the subject.

It is not our aim to give a survey of this literature. A comprehensive account of the development of this subject matter can be read from Jammer's two historicocritical treatises (Jammer, 1966, 1974).

Following Jammer (1974), however, we shall distinguish between two categories of interpretations of the uncertainty principle, which have proved to be most important for the development of quantum mechanics. They are as follows: (1) the nonstatistical interpretation I_1 according to which it is impossible, in principle, to specify the simultaneous values of canonically conjugate variables that describe the behavior of a single physical system; and (2) the statistical interpretation I_2 according to which the product of the standard deviations of two canonically conjugate variables has a lower limit given by $h/4\pi$.

The nonstatistical interpretation I_1 has its roots in Heisenberg's gamma-ray microscope thought experiment, which deals with a single measurement made on a single physical system. The statistical interpretation originated from the above-mentioned work of Popper. This interpretation was elaborated by Henry Margenau (1937) and later developed by the advocates of the ensemble interpretation of quantum mechanics (see, e.g., Ballentine, 1970; Belinfante, 1978). While the nonstatistical interpretation was for many years the dominant interpretation, the statistical interpretation seems to have gained acceptance since 1965 (see Jammer, 1974). Whereas I_1 was generally regarded as being based on arguments involving specific thought experiments, I_2 was held to be established as a straightforward logicomathematical consequence of the very formalism of the theory.

Inquiry for empirical support of the two rival interpretations has not led to a definite choice between them. Though it is certainly true that I_2 enjoys a far better empirical backing than I_1, it is also true that neither of the interpretations is falsified (Jammer, 1974).

In discussing the logical relations between I_1 and I_2 Jammer (1974) argued that I_1 is a logical consequence of I_2 if a certain measurement-theoretical assumption is accepted, i.e., if we assume that every measurement involved in this context is repeatable and, if immediately repeated, yields the same result as its predecessor.

2.5. Our Program. In our approach we treat the uncertainty principle as an axiom of the theory containing essentially the ideas expressed in I_2 above. Similarly, after analyzing Bohr's notion of complementarity, we express its essential content—the relation of mutual exclusion—mathematically, and state the complementarity principle as an axiom of the theory. We then show that the uncertainty principle and the complementarity principle, as mathematized in this work, are mutually independent.

In the light of the results acquired in this work we hold the view that the nonstatistical interpretation I_1 has resulted from a confusion. The arguments which have been given in support of this interpretation are based on thought experiments, which, we think, only support Bohr's viewpoint of complementarity. The confusion derives from the widely accepted view that the uncertainty relations exhibit the mathematical expression of Bohr's notion of complementarity. That this is not the case is shown in this work.

We close this section with a speculation on the possible source of the above-mentioned confusion.

After discussing a lot of thought experiments—a single measurement made on a single physical system—Bohr discovered that the experimental arrangements intended to define position and momentum observables are mutually exclusive. Bohr also liked to give a physical interpretation for the statistical law $\Delta q \Delta p \geqslant h/4\pi$ derived by Heisenberg. For lack of the correct formula, which we shall subsequently give, Bohr ended with the interpretation that "the uncertainty relations are a simple symbolical expression for complementarity" (Bohr, 1927/1978, p. 60).

On the other hand, Heisenberg also used thought experiments in discussing the validity of his statistical law. However, as we already emphasized, the connection between his gamma-ray thought experiment and his statistical law is far from evident. The fact that Heisenberg's thought experiment was very similar to those of Bohr's reveals, as we may learn from Bohr's own careful investigations, mainly the mutual exclusion of the experimental arrangements in question. They only support therefore Bohr's idea of complementarity, and thus the nonstatistical law we give for Bohr's notion of complementarity.

To avoid confusion we should finally like to emphasize that the above discussion does not aim to provide the obviously incorrect statement that the uncertainty relations

$$\mathrm{Var}(A,\phi)\mathrm{Var}(B,\phi) \geqslant \tfrac{1}{4}|(\phi,(AB-BA)\phi)|^2, \qquad \phi \in \mathrm{dom}(AB) \cap \mathrm{dom}(BA)$$

do not admit nonstatistical interpretations, e.g., the one resulting from the acceptance of the propensity interpretation of probability (see, e.g., Popper, 1957, 1959, 1967; Giere, 1973, 1976).

3. ON THE NOTION OF COMPLEMENTARITY

Niels Bohr introduced the notion of complementarity for the first time in 1927 in his classic Como lecture "The Quantum Postulate and the Recent Development of Atomic Theory" to acquire a consistent interpretation of the already then fairly well-established quantum mechanical formalism. During the following 35 years Bohr published a series of essays[3] in which he strove to develop the idea of complementarity into a definite philosophical viewpoint. However, as regards the clarification of the conceptual foundations of the quantum theory, the most important period of

[3]Most of them are collected in the three volumes titled *Atomic Theory and the Description of Nature*, *Atomic Physics and Human Knowledge*, and *Essays 1958–1962 on Atomic Physics and Human Knowledge* originally published in 1934, 1958, and 1963, respectively.

this development seems to begin with his Como lecture in 1927 and to end with his refutation[4] of the Einstein–Podolsky–Rosen criticism in 1935.

In spite of the numerous essays Bohr wrote on the topic, he never gave an explicit definition of the notion of complementarity, or wrote an extensive treatise on the subject.

In this section we shall try to trace the main ideas of Bohr which go under the notion of complementarity. In developing his ideas Bohr investigated "complementary phenomena" not only in physics, but also in other fields like biology, psychology, and sociology. Our account is not meant to be a complete survey of all the features of complementarity. In fact, we are well aware, and also well informed (Petersen, 1968; Jammer, 1974), of the difficulties involved in this kind of task. For our purpose it is enough to study the notion of complementarity in physics only. We thus omit every reference to complementarity in fields outside physics proper.

3.1. Four Kinds of Complementarities. In reading Bohr's writings one may easily form the impression that the notion of complementarity does appear in rather many different connections. However, one can distinguish between four categories of statements which cover most uses of this notion. These are the following: (a) complementarity as a relationship between descriptions, like space–time description and causal description, (b) complementarity as a relationship between elementary physical concepts, like position and momentum, (c) complementarity of the particle picture and the wave picture, and (d) complementarity as a relationship between phenomena demanding mutually exclusive experimental arrangements.

It seems to us that in developing his viewpoint of complementarity Bohr gradually shifted the emphasis from category (a) to category (b), and ultimately "unified" the first three seemingly different notions of complementarity under the one appearing in the category (d).

In fact, in his earlier writings, and especially in his Como lecture, Bohr mainly emphasized "the complementary nature of the space–time description and the claims of causality" (Bohr, 1927/1978, p. 60).

On the other hand, complementarity as a relationship between elementary physical quantities such as position and momentum seems to be foremost in his reply to the Einstein–Podolsky–Rosen paper, where he referred to complementarity as a relation of mutual exclusion of the experimental arrangements intended to define physical concepts such as position and momentum.

[4] Though the Copenhagen school of quantum theory regarded this paper as a refutation of the Einstein–Podolsky–Rosen criticism, it must be admitted that the question is still open.

In picking out category (c) we underline the fact that the very origin of Bohr's notion of complementarity was in his final acceptance of the wave–particle duality of light and matter (Bohr, 1925/1978, 1927/1978; see also Jammer, 1966, and Heisenberg, 1967). According to Bohr, the dualism between the particle and the wave character of light and matter, which is apparent in the well-established Einstein–de Broglie relations, is "avoidable only by means of the viewpoint of complementarity" (Bohr, 1937, p. 294).

In his writings after 1935 Bohr's main concern was the questions of terminology and "dialectics."[5] In these investigations Bohr discussed the above-mentioned types of complementarity under the same heading, namely, in studying phenomena demanding mutually exclusive experimental arrangements. A good example of this tendency is the essay entitled "Discussion with Einstein on Epistemological Problems in Atomic Physics," which is regarded as the clearest exposition of his argumentation (Rosenfeld, 1972; Jammer, 1974).

3.2. Bohr's Early Work on Complementarity. After some introductory remarks Bohr opened his Como lecture with the following statement:

> The quantum theory is characterized by the acknowledgment of a fundamental limitation in the classical physical ideas when applied to atomic phenomena. The situation thus created is of a peculiar nature, since our interpretation of the experimental material rests essentially upon the classical concepts (Bohr, 1927/1978, p. 53).

In this statement with which Bohr introduced the topic, one may *a posteriori* "read" the problem and its solution. The problem is expressed in a paradox, which, according to Heisenberg (1958, p. 44), was the starting point of the Copenhagen interpretation of quantum theory. This paradox, as put in a nutshell by von Weizsäcker (1971, p. 26; see also Heisenberg, 1958, p. 44), reads: "classical physics has been superseded by quantum theory; quantum theory is verified by experiments; experiments must be described in terms of classical physics." The solution of this paradox lies in "the acknowledgment of a fundamental limitation in the classical physical ideas when applied to atomic phenomena," and it was originally approached in two different ways: first through Heisenberg's uncertainty relations, second through Bohr's notion of complementarity (see Heisenberg, 1958, p. 42). A crucial point in Bohr's approach was his very careful examination of the possibilities of definition and observation.

[5]Bohr's term.

Bohr based his argumentation on "the fundamental postulate of the indivisibility of quantum of action" (Bohr, 1929/1978, p. 10). This postulate, which "expresses the essence of the quantum theory," and which "attributes to any atomic process an essential discontinuity, or rather individuality" (Bohr, 1927/1978, p. 53), implies that "any observation of atomic phenomena will involve an interaction with the agency of observation not to be neglected" (Bohr, 1927/1978, p. 54).

When confronting this fact with the classical ideal that "the phenomena concerned may be observed without disturbing them appreciably" (Bohr, 1927/1978, p. 54) Bohr drew some "far-reaching consequences" (Bohr, 1927/1978, p. 54). He wrote:

> On one hand, the definition of the state of a physical system, as ordinarily understood, claims the elimination of all external disturbances. But in that case, according to the quantum postulate, any observation will be impossible, and, above all, the concepts of space and time lose their immediate sense. On the other hand, if in order to make observation possible we permit certain interactions with suitable agencies of measurement, not belonging to the system, an unambiguous definition of the state of the system is naturally no longer possible, and there can be no question of causality in the ordinary sense of the word. The very nature of the quantum theory thus forces us to regard the space–time co-ordination and the claim of causality, the union of which characterizes the classical theories, as complementary but exclusive features of the description, symbolizing the idealization of observation and definition, respectively. (Bohr, 1927/1978, pp. 54 and 55).

In the above statement, where the term "complementary" appeared for the first time, the mutual exclusion of two descriptions, or, the impossibility of the simultaneous use of two descriptions, is apparently put forward. This feature is also clearly expressed in the following statement:

> ...the fundamental postulate of the indivisibility of the quantum of action...forces us to adopt a new mode of description designated as complementary in the sense that any given application of classical concepts precludes the simultaneous use of other classical concepts which in a different connection are equally necessary for the elucidation of the phenomena (Bohr, 1929/1978, p. 10).

It is also put forward in the more general statement that "the word 'complementarity' is used to denote the relation of mutual exclusion characteristic of the quantum theory with regard to the application of the various classical concepts and ideas" (Bohr, 1929/1978, p. 19).

It is important to note that in the above argumentation, as well as throughout his writings, Bohr used expressions like "the claim of causality" in meaning an unambiguous application of the classical energy and/or momentum conservation laws.

The mutual exclusion of two systems of concepts, or two descriptions, is, according to Bohr, most strikingly revealed in the fundamental Einstein–de Broglie relations

$$E = h\nu \quad \text{and} \quad p = h/\lambda$$

which form the basis of the particle description of light and the wave description of matter. In these formulas, as Bohr wrote (Bohr, 1927/1978, p. 58), the two notions of light and also of matter enter into sharp contrast. Indeed, the typical particle concepts energy E and momentum p are here confronted with the typical wave concepts frequency ν and wavelength λ through the universal quantum of action h. Bohr writes: "Here again we are dealing with ...complementary pictures of the phenomena, which only together offer a natural generalization of the classical mode of description" (Bohr, 1927/1978, p. 56).

We hope that the above citations are enough to indicate the early state of Bohr's thinking on the viewpoint of complementarity, which may be summarized as follows: The finite magnitude of the quantum of action implies an inevitable coupling between phenomenon and the agency by which it is observed. This coupling has a consequence that certain modes of descriptions, like space–time description and causal description, which in classical description are united, cannot be used simultaneously in describing situations in which the quantum of action is relevant. Both modes of description being, however, necessary for a full elucidation of the phenomena in question.

3.3. Bohr's 1935 Paper. Bohr's reply in 1935 to the Einstein–Podolsky–Rosen paper marks an important phase in the development of the viewpoint of complementarity. Its importance lies not only in the refutation[6] of the Einstein–Podolsky–Rosen criticism, but also in clarifying the role played by the measuring instruments in observational problems, as well as in bringing into full relief the mutual exclusive character of the experimental arrangements permitting the unambiguous definition of complementary physical quantities. It is also in this connection that Bohr

[6]See footnote 4.

for the first time explicitly referred to the concepts position and momentum as complementary quantities by stating "the mutual exclusive character of any unambiguous use in quantum theory of the concepts of position and momentum" (Bohr, 1935, p. 701).

To clarify his argumentation Bohr devoted in this paper much space to the discussion of the well-known thought experiment 'a particle passing through a slit in a diaphragm'.[7] In discussing this, as well as any other experiment, it is important from the very beginning to specify what purpose a given experimental arrangement is to serve.

In this connection Bohr considered two experimental arrangements: the first allowing one to measure the position of the particle, the second giving one a possibility of measuring the momentum of the particle. The measurement of the position of the particle consists in establishing a rigid connection between the diaphragm, like other parts of the apparatus, and the common support which is to define the space frame of reference. The measurement of the momentum of the particle, which presupposes an unambiguous application of the classical law of conservation of momentum, demands, on the contrary, some movable parts in the arrangement, e.g., the first diaphragm is not rigidly connected with the other parts of the apparatus. As a consequence of the inevitable interaction between object and measuring agencies caused by the finite magnitude of the quantum of action, we are now faced, according to Bohr, in the case of the position measurement, with the impossibility of accurately controlling the displacement of the diaphragm.

In this connection Bohr also emphasized the difference between the two experimental arrangements. In the latter case not only the particle, electron or photon, but also the movable diaphragm belongs to the object under investigation, i.e., in investigating the position of the diaphragm one must take into account the quantum laws governing such measurements. In fact, as Bohr in one of his later articles writes, "the main point here is the distinction between the objects under investigation and the measuring instruments which serve to define, in classical terms, the conditions under which the phenomena appear" (Bohr, 1949, p. 221).

With these two different experimental arrangements Bohr revealed the following:

> ...the mutual exclusive character of any unambiguous use in quantum theory of the concepts of position and momentum (Bohr, 1935, p. 701). In fact, it is only the mutual exclusion of any two experimental procedures, permitting the unambiguous

[7]A refined and still more detailed discussion of this experiment together with some of its modifications is given in Bohr (1949, pp. 218–224).

definition of complementary physical quantities, which provides room for new physical laws, the coexistence of which might at first sight appear irreconcilable with the basic principles of science. It is just this entirely new situation as regards the description of physical phenomena, that the notion of complementarity aims at characterizing" (Bohr, 1935, p. 700).

3.4. Bohr's Late Work on Complementarity. In later years Bohr entered more directly into the questions of terminology and dialectics (see especially Bohr, 1937, 1939, 1948, 1949, 1958a). The necessity of securing unambiguous communication, in which the notion of complementarity is said to have its epistemological roots (Rosenfeld, 1967), is an apparent motive for these investigations.

In these studies, which were of a more philosophical nature, Bohr, in order to avoid certain ambiguities, recommended that the word "phenomenon" be used "to refer exclusively to observations obtained under specified circumstances, including an account of the whole experiment" (Bohr, 1939, p. 24; 1948, p. 317; 1949, p. 238; 1958a, p. 6).

With this terminology Bohr discussed the notion of complementarity in referring to phenomena appearing under mutually exclusive experimental arrangements. In fact, in the above-mentioned articles an expression like "the study of the complementary phenomena demands mutually exclusive experimental arrangements" (Bohr, 1949, p. 211) appears frequently in his expounding of the viewpoint of complementarity.

We close our tracing of the ideas behind the viewpoint of complementarity with the following rather long citation, which, to our judgement, is quite general in nature. Bohr writes:

> In quantum physics evidence about atomic objects obtained by different experimental arrangements exhibits a novel kind of complementary relationship. Indeed, it must be recognized that such evidence which appears contradictory when combination into a single picture is attempted, exhausts all conceivable knowledge about the object. Far from restricting our efforts to put questions to nature in the form of experiments, the notion of complementarity simply characterizes the answers we can receive by such inquiry, whenever the interaction between the measuring instruments and the objects forms an integral part of the phenomena (Bohr, 1958a, p. 4).

3.5. A Definition and a Principle. The apparent dissimilarities between the uses of the term "complementarity" in the given citations from Bohr make it very difficult to form a compact picture of the viewpoint of complementarity, and to speak of *the* notion of complementarity. These dissimilarities appear not only in terminology–complementary descriptions, complementary pictures, complementary physical quantities, complementary phenomena, etc.—but also in the direct use of this notion.

This is especially clear in comparing the statements belonging to categories (a) and (c), respectively. In category (a) two descriptions are discussed which are separated in quantum physics but which are united in classical physics. On the other hand, in category (c) two descriptions are discussed which are united in quantum physics but which are uncombinable in classical physics. More specifically: In case (a) the causal space–time description of classical physics is in quantum physics broken up into two descriptions, namely, into a causal description and into a space–time description, which are denoted as being complementary. In case (c) the two classically irreconcilable pictures (or modes of description), namely, the wave picture and the particle picture, are united in quantum physics into "a complementary mode of description."

Moreover, Bohr's philosophical writings from his later years, in which he struggled to develop a general viewpoint of complementarity, and in which he discussed all the features of complementarity under the same heading, are so vague that they do not readily lend themselves to forming a clear-cut notion of complementarity.

However, we think that a well-established notion of complementarity, for which we can find a mathematical formulation, is necessary in any rational attempt to exploit Bohr's important ideas.

Thus far we have consciously omitted in this section every mention of Bohr's comprehension of Heisenberg's uncertainty relations. It is clear from many connections that for Bohr the uncertainty relations were "a simple symbolical expression for the complementary nature of the space–time description and the claims of causality" (Bohr, 1927/1978, p. 60). (A somewhat more detailed discussion of Bohr's view was given in Section 4.) This view is quite generally accepted, as is evident, e.g., in Rosenfeld's article (Rosenfeld, 1967, p. 121), where he declares that "in atomic physics in the uncertainty relations we can find a mathematical expression for the relation of complementarity," or in Jammer's book (Jammer, 1974, p. 160), where he states that "the operational implications of the notion of complementarity are manifested in the indeterminacy relations." This, however, does not correspond to our view, which we shall subsequently develop.

In looking for the common denominators of the notions of complementarity stated above, we note first that the existence of the quantum of action is presupposed in all of them. Thus we can say that the physical reason for complementarity is Planck's universal quantum of action. The relation of mutual exclusion, resulting from the uncontrollable interaction between the objects and the measuring instruments, is also common to all of them. The relation of mutual exclusion of the experimental arrangements, which permit the unambiguous definition of certain physical quantities, can also easily be given a simple mathematical expression in the Hilbert space formulation of quantum mechanics, as well as in more general axiomatic frameworks.

In our approach we appropriate the following definition.

Definition . Two physical quantities are complementary, if the experimental arrangements permitting the unambiguous definitions of these quantities are mutually exclusive.

Moreover, we adopt the view that the existence of complementary physical quantities is an essential feature of the quantum mechanical description. This attitude we express in the following principle.

The Complementarity Principle. Every physical system possesses, in its quantum mechanical description, complementary physical quantities.

We are well aware of the fact that the given definition together with the stated principle does not exhaust all the ideas of Bohr which go under the notion of complementarity. However, there are, as we shall see, several reasons that defend our choice.

The definition leads quite easily to a mathematical expression, which, by the way, is of a nonstatistical character. Thus the quantum physical law expressed in the principle can be taken into effective use. Moreover, the comparison between the uncertainty principle and the complementarity principle can now be made on solid ground.

Finally, we note that the physical basis of the complementarity principle, namely, Planck's quantum of action, is implicitly taken into account through the definition of complementary physical quantities.

We close this section with a short bibliographical note.

The literature dealing with complementarity is quite extensive, and also very divergent. As an example of two entirely opposing valuations of Bohr's viewpoint of complementarity we may refer on the one hand to Rosenfeld, who writes that the solution of the interpretation problem of quantum mechanics "received its final formulation from Niels Bohr; the new logical instrument which was created by Bohr is called complementarity" (Rosenfeld, 1961, p. 385), and on the other hand to Popper, who writes that "the principle of complementarity has remained completely sterile within physics. In twenty-seven years it has produced nothing except some philosophical discussion..." (Popper, 1963, p. 101).

In order to keep our account of Bohr's notion of complementarity at reasonable length we have mainly resorted to Bohr's original writings.

However, there are some studies on complementarity which, we think, are of great help in any attempt to post oneself up on Bohr's philosophy. The two historicocritical books of Jammer (1966, 1974) as well as the philosophical investigation of Petersen (1968) earn special mention for their clear exposition of Bohr's ideas. Also the writings of Grünbaum (1957), Holton (1970), Hooker (1972), Petersen (1963), Rosenfeld (1961, 1963, 1967, 1971, 1972), and von Weizsäcker (1955, 1971, 1973a), provide very important contributions to the understanding of Bohr's philosophy.

4. SOME PROPERTIES OF POSITION AND MOMENTUM OBSERVABLES IN THE HILBERT SPACE FORMULATION OF QUANTUM MECHANICS

In the Hilbert space formulation of quantum mechanics the canonically conjugate position and momentum operators Q and P have the following important properties:

$$\mathrm{Var}(Q,\phi)\,\mathrm{Var}(P,\phi) \geqslant (h/4\pi)^2 \quad \text{for all } \phi \text{ in } \mathrm{dom}(QP)\cap\mathrm{dom}(PQ) \tag{4.1}$$

$$P^Q(E)\wedge P^P(F)=0 \quad \text{for all bounded } E \text{ and } F \text{ in } B(R) \tag{4.2}$$

Here $\mathrm{Var}(Q,\phi)/\mathrm{Var}(P,\phi)/$ denotes the variance of $Q/P/$ in the state ϕ, $P^Q/P^P/$ denotes the spectral measure of $Q/P/$, and $B(R)$ denotes the set of all Borel subsets of the real line R.

The first of these relations emphasizes the noncommutativity of Q and P. Elegant derivations for this inequality, the uncertainty relation for Q and P, can be read in many texts (see, e.g., Jauch, 1968; Prugovecki, 1971; Packel, 1974). A popular "textbook interpretation" of this inequality is found in the original interpretation of Heisenberg: The inequality (4.1) gives a limitation to the accuracy in which position and momentum observables can simultaneously be measured. That is, if we increase the accuracy in measuring, e.g., the position of our system, the accuracy in measuring its momentum inevitably decreases in such a way that the product of their suitably defined "uncertainties" is always greater than or equal to the constant $h/4\pi$ (see, e.g., Heisenberg, 1949; von Neumann, 1955; Jauch, 1968). This and other interpretations of the uncertainty principle were discussed in Section 2 of this work.

The second of these relations is a consequence of the unitary equivalence of Q and P given by the Fourier–Plancherel transformation. We shall give a derivation for this result in this section. The physical content of the equation (4.2) is the following: There are no states of a physical system such that both the position and the momentum observables are contained in given finite intervals. That is, there exists no experimental arrangement by which the expressions "the position observable has a value in $[x_0, x_0 + \Delta x]$" and "the momentum observable has a value in $[p_0, p_0 + \Delta p]$" can simultaneously be verified. We emphasize that this is the case not only for the choice $\Delta x \Delta p < h$, but for any finite Δx and Δp. Equation (4.2) takes into account the fact that the experimental arrangements intended to define position and momentum observables are mutually exclusive. Thus it emphasizes the complementary nature of position and momentum observables.

4.1. A Theorem. We shall now proceed by giving a Hilbert space derivation for the above-mentioned property (4.2) of position and momentum observables.

To this end we consider the wave mechanical canonical position and momentum operators q and p for a particle confined to move in one dimension. The Hilbert space H of the physical system concerned is the Lebesgue space $L_2(R)$.[8]

The position operator q is defined as the operator with domain D_q:

$$D_q = \left\{ \phi \in L_2(R) : \int_R x^2 |\phi(x)|^2 \, dx < +\infty \right\}$$

and which acts on D_q in the following way:

$$(q\phi)(x) = x\phi(x) \quad \text{for all } x \text{ in } R \text{ and } \phi \text{ in } D_q$$

The momentum operator p conjugate to q is defined by

$$p = \frac{h}{2\pi} F^{-1} q F, \qquad D_p = F^{-1} D_q$$

where F is the Fourier–Plancherel operator of $L_2(R)$. The operator F is defined pointwise by the equation

$$(F\phi)(x) = \lim_{T \to +\infty} (2\pi)^{-1/2} \int_{-T}^{+T} \phi(t) e^{-ixt} \, dt, \qquad \forall x \in R$$

which is valid for all (continuously differentiable) ϕ in $L_2(R)$. The operator

[8]For standard definitions and results appearing in the following we refer to Prugovecki (1971).

F is unitary and the operators q and p are self-adjoint. Moreover, one can easily verify that p is nothing else but the differential operator $(h/2\pi i)(d/dx)$.

The spectral measure P^q of q is given by the formula

$$P^q(E)\phi = \chi_E\phi \quad \text{for all } E \text{ in } B(R) \text{ and } \phi \text{ in } L_2(R)$$

where χ_E denotes the characteristic function of the set E.

Because of the unitary equivalence of q and p the corresponding spectral measures P^q and P^p are unitarily equivalent too. Hence we have

$$P^p(E) = \frac{h}{2\pi} F^{-1} P^q(E) F \quad \text{for all } E \text{ in } B(R)$$

Now we prove a theorem which shows that there are no states in the Hilbert space $L_2(R)$ such that both q and p are contained in given finite intervals.

Theorem 4.1. For any bounded Borel sets E and F in $B(R)$ we have

$$P^q(E) \wedge P^p(F) = 0$$

Proof. For convenience we use in the proof atomic units in which $h/2\pi = 1$. Suppose that there exist two finite intervals I and J of R such that $P^q(I) \wedge P^p(J) \neq 0$. This implies the existence of a nonzero vector ϕ in $L_2(R)$ for which

$$P^q(I)\phi = \phi, \quad \text{i.e., } \phi \in P^q(I)(L_2(R))$$

$$P^p(J)\phi = \phi, \quad \text{i.e., } \phi \in P^p(J)(L_2(R))$$

Now $P^q(I) = \chi_I$ and $P^p(J) = F^{-1}P^q(J)F$ so that the above equations can be written in the form

$$\chi_I\phi = \phi \quad \text{and} \quad \chi_J\tilde{\phi} = \tilde{\phi}$$

where $\tilde{\phi} = F\phi$. This means that for every t in R

$$\begin{aligned}\tilde{\phi}(t) = (F\phi)(t) &= (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \phi(x) e^{-ixt}\,dx \\ &= (2\pi)^{-1/2}\int_{-\infty}^{+\infty} (\chi_I\phi)(x) e^{-ixt}\,dx \\ &= (2\pi)^{-1/2}\int_I \phi(x) e^{-ixt}\,dx = (\chi_J\tilde{\phi})(t)\end{aligned}$$

so that

$$\int_I \phi(x)e^{-ixt}\,dx=0, \qquad \forall t \notin J=[t_1, t_2]$$

Thus we have

$$\int_{-\infty}^{t_1} \tilde{\phi}(t)\,dt=0 \quad \text{and} \quad \int_{t_2}^{+\infty} \tilde{\phi}(t)\,dt=0$$

which give after the t integrations the equalities

$$\text{(a)} \qquad \int_I x^{-1}\phi(x)e^{-ixt_1}\,dx=\int_I x^{-1}\phi(x)\Big(\lim_{t\to-\infty} e^{-ixt}\Big)\,dx \qquad (x\neq 0)$$

$$\text{(b)} \qquad \int_I x^{-1}\phi(x)e^{-ixt_2}\,dx=\int_I x^{-1}\phi(x)\Big(\lim_{t\to+\infty} e^{-ixt}\Big)\,dx \qquad (x\neq 0)$$

If there exists an $x>0$ such that $\phi(x)\neq 0$ (and hence $x\in I$), then by the continuity of ϕ there exists a neighborhood of x in which ϕ is nonzero. In this case equation (a) cannot be valid, since the left-hand side is bounded and the right-hand side is unbounded. Similarly, if there exists an $x<0$ such that $\phi(x)\neq 0$ (and hence $x\in I$), then the equation (b) cannot be valid. So we conclude that $\phi=0$, which is in contradiction with our assumption. This completes the proof. □

The wave mechanical position and momentum operators q and p satisfy the well-known operator equation

$$qp-pq=\frac{ih}{2\pi}I \tag{4.3a}$$

with a domain which is dense in $L_2(R)$.

On the other hand, the general problem of finding all the pairs of self-adjoint operators Q and P on a Hilbert space H which satisfy the equation

$$QP-PQ=\frac{ih}{2\pi}I \tag{4.3b}$$

on some "sufficiently large" domain $D\subset H$ has the well-known solution: (Q, P) is a Schrödinger couple or a direct sum of Schrödinger couples (see, e.g., Putnam, 1967; Packel, 1974). In this solution the term "sufficiently large" receives its meaning in the density of $(P+I)(Q+I)D$ or $(Q+I)$

$(P+I)D$ in H. A Schrödinger couple is a pair (Q, P) of self-adjoint operators on H such that $Q=UqU^{-1}$ and $P=UpU^{-1}$ for some unitary $U: L_2(R) \rightarrow H$.

Let P^Q and P^P be the spectral measures of the Schrödinger couple (Q, P). Because $Q=UqU^{-1}$ and $P=UpU^{-1}$ we have for any $n \in N$

$$(P^Q(E)P^P(F))^n = U(P^q(E)P^p(F))^n U^{-1}$$

The algebraic construction for the lattice meet of any two orthogonal projections P and R in H is given by (von Neumann, 1950)

$$P \wedge R = s - \lim_{n \rightarrow \infty} (PR)^n$$

which shows that

$$P^Q(E) \wedge P^P(F) = U(P^q(E) \wedge P^p(F))U^{-1}$$

We conclude that theorem 4.1 holds for any Schrödinger couple, and hence equation (4.2) is established.

4.2. Some Remarks. Some remarks are now called for.

First, we should recall the important distinction between the Schrödinger (or the Weyl) couples and the Heisenberg couples (see Garrison and Wong, 1970). A Schrödinger couple consists of any two operators Q and P which are unitarily equivalent, respectively, to the position and the momentum operators q and p for a free particle in one dimension. A Heisenberg couple (Q, P) consists of two densely defined self-adjoint operators together with a dense subspace $D \subset H$ on which their commutator $QP-PQ$ is defined and satisfies the relation

$$(QP-PQ)\phi = \frac{ih}{2\pi} I\phi \quad \text{for all } \phi \text{ in } D$$

It is clear that a Schrödinger couple is a Heisenberg couple, but the converse is not necessarily true. In fact, an example of a Heisenberg couple which is not a Schrödinger couple is provided by the problem of a particle in a onedimensional box of unit length. In the usual way defined canonical position and momentum operators give such an example (see Garrison and Wong, 1970; see also Section 6.3 in our work).

So we emphasize that Theorem 4.1 may not be true for a Heisenberg couple which is not a Schrödinger couple.

With this result it is interesting to confront the following statement of Bohr which he made in studying the behavior of the electron in the atom with wave mechanical terms. Bohr arrived at the conclusion that if a measurement of an electron's coordinate is possible at all, the electron must be practically free (Bohr, 1927/1978, pp. 78 and 79; see also Heisenberg, 1930/1949, p. 19).

Second, the fact that there are no states of a physical system such that both the position and the momentum observables are contained in given finite intervals seems to be well known, though perhaps not often enough explicitly stated. In fact, in discussing our theorem with S. Bugajski, he kindly informed us of a work of Jauch in which Jauch, while discussing the problem of the joint probability distribution for noncompatible observables, paid attention to this result (Jauch, 1976). Also M. Jammer kindly showed us one of his recent papers (Jammer, 1978) in which he explicitly mentioned this fact.

Third, we shall rediscuss our interpretation for relation (4.2) above. Let us begin by discussing briefly the problem of defining position and momentum observables in quantum mechanics.

We recall first the following mathematical results:

$$\langle\{[a,b]:a,b\in R\}\rangle = B(R) \tag{4.4}$$

$$\langle\{P^Q[a,b]:a,b\in R\}\rangle = \{P^Q(E):E\in B(R)\} \tag{4.5}$$

$$\langle\{P^P[a,b]:a,b\in R\}\rangle = \{P^P(E):E\in B(R)\} \tag{4.6}$$

$$P^Q[a,b]\wedge P^P[c,d]=0, \qquad \forall a,b,c,d\in R \tag{4.7}$$

Here $\langle\{\cdots\}\rangle$ denotes the Boolean σ algebra generated by $\{\cdots\}$.

In physical terms to define the position of a particle it is enough to specify all the yes–no experiments corresponding to projectors $P^Q[a,b]$, a, $b\in R$. An experimental arrangement corresponding to the yes–no experiment "$P^Q[a,b]$" is given by a diaphragm with a slit of width $[a,b]$ rigidly connected to a common support defining the space frame of reference.

Similarly, to define the momentum of a particle it is enough to specify all the yes–no experiments corresponding to projectors $P^P[c,d]$, c, $d\in R$, and to give experimental arrangements allowing one to perform these yes–no experiments.

Now, what Bohr has clearly shown is that the experimental arrangements needed for the unambiguous definition of the position and the momentum of the particle are mutually exclusive, i.e., all the yes–no experiments "$P^Q[a,b]$" and "$P^P[c,d]$" are mutually exclusive. This situation corresponds exactly to the above Hilbert space result (4.7).

Bohr, however, for lack of the correct formula (4.7), interpreted the uncertainty relations, statistical relations

$$\mathrm{Var}(Q,\phi)\,\mathrm{Var}(P,\phi) \geqslant (h/4\pi)^2, \qquad \forall\phi\in\mathrm{dom}(QP)\cap\mathrm{dom}(PQ) \tag{4.8}$$

"as a simple symbolical expression for the notion of complementarity." This widely accepted view, which we think is wrong, has caused much confusion (see, e.g., Section 7.1).

We interpret Theorem 4.1 as a mathematical expression for Bohr's notion of complementarity. So we conclude that complementarity is also, as the uncertainty relations are, a mathematical consequence of the Hilbert space formulation of quantum mechanics. Thus it is not only "an extraneous interpretative addition to it," as, e.g., Jammer argues (Jammer, 1974, p. 60).

In discussing our interpretation of Theorem 4.1 with Professor Jammer, he suggested this result should be called a "strong version of the indeterminacy principle" (Jammer, 1978). But a strong version of the indeterminacy principle (4.7) should imply the indeterminacy principle (4.8). However, we shall give some arguments in favor of the independence of the relations (4.7) and (4.8).

5. THE (O, S, p) THEORY—A SHORT SKETCH

In the (O, S, p) formulation of a general physical theory it is assumed that with each physical system F one can associate the set of all observables O of F and the set of all states S of F, and a function $p: O\times S\times B(R)\to[0,1]$, where $B(R)$ is the set of all Borel subsets of the real line R. The function p is interpreted through its range: The number $p(A,\alpha,E)$ gives the probability that a measurement of A in the state α will yield a result in E. Thus the function p, which is called the probability function of the system F, gives the connection between the theory and the experiment. The given interpretation for the function p requires that for each fixed A in O and each fixed α in S the set function $B(R)\to[0,1]$, $E\to p(A,\alpha,E)$ is a probability measure on $B(R)$.

We shall not go into details in sketching the theory which arises in a natural way from this approach.

Referring to the works of Mackey (1963) and Maczynski (1967, 1973) we take for granted that with each physical system F, with probability function p, there is associated in a very natural way an orthomodular,

σ-orthocomplemented partially ordered set L, the logic of p, whose members are called propositions (denoted by a) or experimental propositions (denoted by $|A, E|$) or experimental functions (denoted by $f_{|A,E|}$). Each observable A in O determines a unique L-valued measure $\mu_A: B(R)\to L$, $E\to\mu_A(E)$, and each state α in S determines a unique probability measure on $L m_\alpha: L\to[0,1]$, $a\to m_\alpha(a)$. The family of L-valued measures corresponding to all observables $\{\mu_A: A\in O\}$ is surjective, i.e., for any a in L there is an A in O and an E in $B(R)$ such that $a=\mu_A(E)$. The family of probability measures corresponding to all states $\{m_\alpha: \alpha\in S\}$ is full, i.e., order determining. For each A in O, each α in S, and each E in $B(R)$, we have

$$p(A,\alpha,E)=m_\alpha\circ\mu_A(E) \tag{5.1}$$

The number $p(A,\alpha,E)$ is said to give the probability that a measurement of A in the state α will yield a result in E. In posing some properties on the function $p: O\times S\times B(R)\to[0,1]$ we end with the given formal structure of what is sometimes termed "quantum probability theory." The question of the meaning of probability is, however, thus far left open. This question is definitely of great importance. Even the various interpretations of the quantum theory essentially arise out of the differences in the interpretation of the probabilities predicted by the theory (see, e.g., Popper, 1967; Strauss, 1973; Jammer, 1974; Jauch, 1976; and von Weizsäcker, 1973).

A commitment to any specific interpretation of probability, and thus of the quantum theory, is irrelevant for the present work. The most popular interpretation of quantum theory is the so-called ensemble interpretation (cf. Section 2.4) which results from adopting the relative frequency interpretation of probability. According to the relative frequency interpretation of probability the probability of an event is the limit of its relative frequency in the long run. For detail exposition of this interpretation we refer to B. C. van Fraassen (1977).

The above-sketched (O, S, p) formulation of a general physical theory belongs to the quantum logic approach to axiomatic quantum mechanics. The reason we contented ourselves only with a summary of this approach, and completely neglected the other approaches, is threefold. First, the given structure, which is essentially what we need subsequently, is very well known and quite generally accepted. Second, we gave in our licenciate thesis (Lahti, 1976b) a fairly extensive survey on the quantum logic approach. Last, but not least, the existing literature contains many excellent survey articles, and also some books, on this as well as on other contemporary approaches to axiomatic quantum mechanics. We need mention only Mackey (1963), Jauch (1968), Varadarajan (1968), Gudder (1970b, 1977), Greechie and Gudder (1973), Jammer (1974), Piron (1976), and Beltrametti and Cassinelli (1976). We note also the forthcoming book *The Logic of Quantum Mechanics* of Beltrametti and Cassinelli.[9]

6. THE AXIOM OF HEISENBERG

6.1. The Need for an Axiom. In the Hilbert space formulation of quantum mechanics for any two self-adjoint operators A and B and for any state ϕ in $\mathrm{dom}(AB) \cap \mathrm{dom}(BA)$ the following inequality holds

$$\mathrm{Var}(A,\phi)\,\mathrm{Var}(B,\phi) \geqslant \tfrac{1}{4}|(\phi,(AB-BA)\phi)|^2 \tag{6.1}$$

The left-hand side of this inequality can easily be formulated in quantum logics, too. For example, it can be given in terms of experimental functions, i.e., mappings $f_{|A,E|}: S \to [0,1]$, $\alpha \to f_{|A,E|}(\alpha) = p(A,\alpha,E)$ (Maczynski, 1973; see also Lahti, 1979).

The right-hand side of the above inequality measures, intuitively speaking, the noncommutativity of the observables A and B. An analogy for this can be given in terms of what is usually called ideal, pure, first-kind measurements (see, e.g., Beltrametti and Cassinelli, 1976). Indeed, using state transformations corresponding to this kind of measurements we may define for every a and b in L

$$\Omega_a \circ \Omega_b - \Omega_b \circ \Omega_a \tag{6.2}$$

as

$$(\Omega_a \circ \Omega_b)(\alpha)(c) - (\Omega_b \circ \Omega_a)(\alpha)(c) \tag{6.3}$$

for every α in S and c in L. Such defined quantity (6.2) can obviously be used to "measure" the commutativity (compatability) of a and b. In fact, the real number (6.3) is zero for every α in S and c in L if and only if a and b commute (are compatible).

In studying the connection between experimental functions $f_{|A,E|}: S \to [0,1]$ and state transformations $\Omega_{|A,E|}: S \to S$ it may be possible to establish a relation like (6.1).

Thus far we do not, however, have any generalizations of the uncertainty principle for quantum logics (see also Gudder, 1978). The difficulty

[9] This book will appear in the *Encyclopedia of Mathematics and Applications*, G. C. Rota, ed., Addison-Wesley, Reading, Massachusetts.

in generalizing the uncertainty principle for quantum logics lies in the first place in the fact that in quantum logics we do not have any analogy for the Hilbert space Schwarz inequality, which is essential in the derivation of the inequality (6.1).

In order to utilize the uncertainty principle in quantum logics we thus have to express it as an axiom of the theory. The idea of treating the uncertainty principle as an axiom of the theory is almost as old as the principle itself. Indeed, in his Chicago lectures Heisenberg writes

> ...in many cases it is impossible to obtain an exact determination of the simultaneous values of two variables, but rather that there is a lower limit to the accuracy with which they can be known...this lower limit to the accuracy with which certain variables can be known simultaneously may be postulated as a law of nature...(Heisenberg, 1930/1949, p. 3).

With the above considerations in mind we pose the following axiom, which we name after Heisenberg.

The Axiom of Heisenberg. There exist at least two observables A and B in O such that in every state α in S (for which the quantities in question are well defined) the product of their variance is greater than or equal to a given positive constant, say, h.

It is worth while emphasizing that the interpretation of the probabilistic concept "variance" appearing in the axiom is, in this connection, left open for any consistent interpretation. (See also the corresponding notes on p. 800 and 816.)

As already mentioned, in postulating the uncertainty principle as an axiom of the theory we are in a position to take this important principle into effective use in studying quantum logics. Moreover, in this way we can explicitly introduce the quantum of action into quantum logics. Though we have not specified the constant h above, we could equally well postulate this constant to be $(h/4\pi)^2$. This is especially important, because in usual formulations of quantum logics the root of the quantum theory, the quantum of action h, does not appear at all in the theory.

Before going further we shall make some remarks on the possibilities of calculating the constant h. First, we could use the quantity (6.2) above for this purpose. This approach, however, does not readily lend itself to any concrete calculations. A more specific method is suggested by Maczynski (1980). He defines a real-valued function ρ, which assigns to every pair of experimental functions (or propositions) a and b a number $0 \leqslant \rho(a,b) \leqslant 1$ with the property: $\rho(a,b)=0$ if and only if a and b commute. Maczynski has given an explicit, and in principle easily calculable, form for the function ρ (Maczynski, 1978).

6.2. Preliminary Considerations. Each observable A in O and each state α in S defines a probability measure $m_\alpha \circ \mu_A : B(R) \to [0,1]$, which describes the distribution of A in the state α. This means that we can define the expectation value of A in the state α in a standard, i.e., Kolmogorovian way as the integral

$$\exp(A, \alpha) = \int_R i \, dm_\alpha \circ \mu_A \tag{6.4}$$

provided that it exists (Loeve, 1955). Here, of course, i denotes the identity function on R.

Let S_A^E denote the set of all states α in S for which $\exp(A, \alpha)$ exists and is finite.

Now we can define the variance of A in the state α as the quantity

$$\mathrm{Var}(A, \alpha) = \exp(A^2, \alpha) - \exp(A, \alpha)^2 \tag{6.5}$$

provided that the integrals in question exist.

Let S_A^V denote the set of all states α in S for which $\mathrm{Var}(A, \alpha)$ exists and is finite. Note that S_A^V is contained in S_A^E, but in general not conversely.

In general for any two observables A and B in O one of the following three possibilities holds:

$$\mathrm{Var}(A, \alpha)\,\mathrm{Var}(B, \alpha) = 0 \quad \text{for all } \alpha \text{ in } S \tag{6.6}$$

$$(\forall \varepsilon > 0)(\exists \alpha \in S)(\mathrm{Var}(A, \alpha)\,\mathrm{Var}(B, \alpha) < \varepsilon) \tag{6.7}$$

$$(\exists \varepsilon > 0)(\forall \alpha \in S)(\mathrm{Var}(A, \alpha)\,\mathrm{Var}(B, \alpha) \geqslant \varepsilon) \tag{6.8}$$

We note that (6.6) is a special case of (6.7). However, we prefer to state it separately.

In discussing the above three possibilities we need some spectral concepts, which we recall in the following (see, e.g., Gudder, 1970b, Varadarajan, 1968):

The spectrum of A: $\sigma(A) = \cap(E : E \subset R \text{ closed}, \mu_A(E) = 1)$.

The point spectrum of A : $\sigma_P(A) = \{\lambda \in R : \mu_A(\{\lambda\}) \neq 0\}$.

The continuous spectrum of A : $\sigma_c(A) = \sigma(A) \setminus \sigma_p(A)$.

A state α in S is an eigenstate of A if there exists a real number λ such that $m_\alpha(\mu_A(\{\lambda\})) = 1$. In this case λ is called the eigenvalue of A corresponding to the eigenstate α. Note that an eigenvalue may correspond to many eigenstates, but an eigenstate can have only one eigenvalue. Clearly, eigenvalues of A are contained in the point spectrum of A.

A is bounded if $\sigma(A) \subset R$ is bounded. Note that A is bounded if and only if $S_A^E = S$ (Gudder, 1970b). Of course, if $S_A^V = S$, then A is bounded. On the other hand, if A is bounded, i.e., $S_A^E = S$, then one easily verifies that $\mathrm{Var}(A, \alpha)$ exists and is finite in every state α in S. Hence we also have: A is bounded if and only if $S_A^V = S$.

We pose the following extra assumption on S:

$$\text{For every } a \text{ in } L,\ a \neq 0, \text{ there exists a state } \alpha \text{ in } S \text{ such that } m_\alpha(a) = 1. \tag{6.9}$$

That is, we suppose that for every nonzero experimental proposition there exists an experimental arrangement, i.e., a preparation of state of F, which would verify the proposition with certainty.

Our motivations for the above assumption are the following: A proposition a in L is, by definition, nonzero if and only if there exists a state α in S such that $m_\alpha(a) \neq 0$. This means—in accordance with the notion of probability—that the proposition a is "possible." In order for an event (or proposition) to be "possible" there must be some conditions under which this event is "actual." Adopting the view that observation (or measurement) selects from all possible events the actual one which has taken place (Heisenberg, 1958), we are led to the assumption (6.9), which is usually called the projection postulate.[10] (For a detailed analysis of projection postulates in quantum logics see K. Bugajska and S. Bugajski 1973).

An immediate consequence of the assumption (6.9) is that the point spectrum of A contains all the eignevalues of A and only them.

We shall proceed by discussing $\mathrm{Var}(A, \alpha)$ in some special cases.

1. Let A be a constant observable, i.e., an observable with range $\{0, 1\}$ in L. In this case $\exp(A, \alpha) = 0$ and $\mathrm{Var}(A, \alpha) = 0$ for every state α in S.

2. Let A be an elementary observable with range $\{0, a, a^\perp, 1\}$, and with spectrum $\{0, 1\}$ in R. In this case we have for all α in S $\exp(A, \alpha) = \int_R id\, m_\alpha \circ \mu_A = m_\alpha(a)$ and $\mathrm{Var}(A, \alpha) = m_\alpha(a) - m_\alpha(a)^2$. So $\mathrm{Var}(A, \alpha) = 0$ if and only if $m_\alpha(a) = 0$ or $m_\alpha(a) = 1$.

3. An elementary observable is a special case of the discrete observable (Gudder, 1970b). Let (a_i) be a sequence of mutually disjoint elements of L such that $\bigvee a_i = 1$, and let (λ_i) be a sequence of distinct real numbers. The map $\mu: B(R) \to L$, $E \to \mu(E) = \bigvee\{a_i : \lambda_i \in E\}$ is an observable with spectrum $\sigma(A) = \sigma_p(A) = \{\lambda_i\}$. In this case $\exp(A, \alpha) = \Sigma \lambda_i m_\alpha(a_i)$, and $\mathrm{Var}(A, \alpha) = \Sigma \lambda_i^2 m_\alpha(a_i) - [\Sigma \lambda_i m_\alpha(a_i)]^2$ for every state α in S for which the quantities in question are well defined.

[10] This is also known as the weak form of the projection postulate, and it is usually referred to as the sufficiency of the state systems S.

4. Let α in S be an eigenstate of A and λ the corresponding eigenvalue of A. In this state α we have $\exp(A, \alpha)=\lambda$ and $\mathrm{Var}(A, \alpha)=0$.

5. In fact, $\mathrm{Var}(A, \alpha)=0$ if and only if the probability measure $m_\alpha \circ \mu_A$ is concentrated on a point, i.e., A is a constant observable or α is an eigenstate of A.

6.3. Three Lemmas. In this section we shall prove the three lemmas which help us to characterize pairs of observables satisfying the axiom of Heisenberg. The sufficiency of the state system S is assumed throughout this section. We begin by proving a lemma which corresponds to the intuitive view that there exists no lower limit to the accuracy in measuring single physical quantities (cf., e.g., Heisenberg, 1930/1949).

> *Lemma 6.1.* Let A be any observable in O. For each $\varepsilon>0$ there exists a state α in S such that $\mathrm{Var}(A, \alpha)<\varepsilon$.

Proof. Let $\sigma(A)$ be the spectrum of A and let λ by any fixed element in $\sigma(A)$. In general $\lambda \in \sigma(A)$ if and only if for each $\eta \in R$, $\eta>0$, $\mu_A((\lambda-\eta, \lambda+\eta)) \neq 0$ (Gudder, 1970b). Without any loss of generality we can suppose that $0<\lambda-\eta<\lambda+\eta<\infty$. Let α be a state in S such that $m_\alpha(\mu_A((\lambda-\eta, \lambda+\eta)))=1$. In this state α, which depends on η, we have

$$\mathrm{Var}(A, \alpha)=\int i^2\, dm_\alpha \circ \mu_A-\left(\int i\, dm_\alpha \circ \mu_A\right)^2 \leqslant(\lambda+\eta)^2-(\lambda-\eta)^2=4 \lambda \eta$$

which is less than the given positive real number ε with the choice $\eta<\varepsilon / 4 \lambda$. In the case $\lambda=0$ we immediately get $\mathrm{Var}(A, \alpha) \leqslant \eta^2$. □

As a consequence of this lemma and the note on p. 820 we have the following result.

> *Lemma 6.2.* Let A and B be two observables in O. If one of them is bounded, then
>
> $$(\forall \varepsilon>0)(\exists \alpha \in S)(\mathrm{Var}(A, \alpha) \mathrm{Var}(B, \alpha)>\varepsilon)$$

Proof. Suppose that A is bounded, so that $S_A^V=\mathrm{S}$. Let M be a positive real number such that $\mathrm{Var}(A, \alpha) \leqslant M$ for all α in S. Applying the above lemma for B we can find for any positive real number ε a state α in S such that $\mathrm{Var}(A, \alpha) \mathrm{Var}(B, \alpha) \leqslant M \cdot \mathrm{Var}(B, \alpha)<\varepsilon$. □

We emphasize that in lemma 6.2 the compatibility of the observables A and B is not assumed. In other words, lemma 6.2 is true even for noncompatible observables. As an illustration of this we give a Hilbert space description of the problem of a particle in a one-dimensional box of unit length. (For another example see the note below lemma 6.3.) Choosing the Hilbert space H associated with the system to be the Lebesgue space $L_2(0,1)$ we define the canonical position and momentum operators Q and P by

$$(Q\phi)(x)=x\phi(x) \quad \text{and} \quad (P\phi)(x)=-\frac{ih}{2\pi}\frac{d\phi(x)}{dx}\equiv-\frac{ih}{2\pi}\phi'(x)$$

with domains $\operatorname{dom}(Q)=H$ and $\operatorname{dom}(P)=\{\phi\in H\colon \phi$ is absolutely continuous, $\phi'\in H$, $\phi(0)=\phi(1)\}$, respectively. Thus the operators Q and P are densely defined and self-adjoint. Moreover, they satisfy the equation $QP-PQ=(ih/2\pi)I$ in $\operatorname{dom}(QP)\cap\operatorname{dom}(PQ)$ which is dense in H. But now the position observable Q is bounded, with spectrum $\sigma(Q)=[0,1]$. Thus for every $\varepsilon>0$ we can find a state ϕ in H such that $\operatorname{Var}(Q,\phi)\operatorname{Var}(P,\phi)<\varepsilon$. For example, any eigenstate ϕ_n of P will do that. We emphasize that this result is in no contradiction with the inequality $\operatorname{Var}(Q,\phi)\operatorname{Var}(P,\phi)\geqslant(h/4\pi)^2$, which is valid only for those states which are in $\operatorname{dom}(QP)\cap\operatorname{dom}(PQ)$.

We note that for any state α in $S_A^V\cap S_B^V$ $\operatorname{Var}(A,\alpha)\operatorname{Var}(B,\alpha)=0$ if and only if either A or B is a constant observable, or α is an eigenstate of A or B.

Let A and B be compatible observables. This is the case if and only if there is an observable C and Borel functions f and g such that $A=f(C)$ and $B=g(C)$ (see, e.g., Gudder, 1970b; Varadarajan, 1968). Now we have for any state α in S

$$\int i\,dm_\alpha\circ\mu_A=\int f\,dm_\alpha\circ\mu_C \tag{6.10}$$

$$\int i^2\,dm_\alpha\circ\mu_A=\int f^2\,dm_\alpha\circ\mu_C \tag{6.11}$$

in the sense that if one of the integrals appearing in (6.10) [respectively, in (6.11)] exists then both of them exist, and in this case they are equal (Halmos, 1950).

Lemma 6.3. If A and B are compatible observables, then

$$(\forall\varepsilon>0)(\exists\alpha\in S)(\operatorname{Var}(A,\alpha)\operatorname{Var}(B,\alpha)<\varepsilon)$$

Proof. Let A and B be two compatible and nonconstant observables.

Thus there exist, at least, two bounded Borel sets $E \subsetneq \sigma(A)$ and $F \subsetneq \sigma(B)$ in $B(R)$ such that (cf. the proof of Theorem 7.1.)

$$\mu_A(E) \wedge \mu_B(F) \neq 0$$

Let $E \subset I = [\lambda_1, \lambda_2]$ and $F \subset J = [\eta_1, \eta_2]$. Thus we also have

$$\mu_A(I) \wedge \mu_B(J) \neq 0$$

Because of the sufficiency of S there is a state α in S such that

$$m_\alpha(\mu_A(I) \wedge \mu_B(J)) = 1$$

and also $m_\alpha(\mu_A(I)) = 1$ and $m_\alpha(\mu_B(J)) = 1$. Without any loss of generality we can assume the $0 < \lambda_1 < \lambda_2$ and $0 < \eta_1 < \eta_2$. In the above state α we now have

$$(*) \quad \operatorname{Var}(A, \alpha) \operatorname{Var}(B, \alpha) \leqslant (\lambda_2^2 - \lambda_1^2)(\eta_2^2 - \eta_1^2) =_d fh(\lambda_1, \lambda_2) h(\eta_1, \eta_2)$$

We note that the function h defined above is continuous with respect to both variables, and it is also bounded. The sufficiency of S and the continuity of h are now enough to prove the lemma. The method is simply the following. If the right-hand side of $(*)$ is not less than the given $\varepsilon > 0$ write $[\lambda_1, \lambda_2) = [\lambda_1, \lambda] \cup [\lambda, \lambda_2]$ with $\lambda_1 < \lambda < \lambda_2$ [e.g., $\lambda = \lambda_1 + \frac{1}{2}(\lambda_2 - \lambda_1)$]. Because $0 \neq \mu_A(I) \wedge \mu_B(J) = (\mu_A([\lambda_1, \lambda)) \vee \mu_A([\lambda, \lambda_2])) \wedge \mu_B(J) = (\mu_A([\lambda_1, \lambda)) \wedge \mu_B(J)) \vee (\mu_A([\lambda, \lambda_2]) \wedge \mu_B(J))$ we may assume that

$$\mu_A([\lambda_1, \lambda)) \wedge \mu_B(J) \neq 0$$

Because of the sufficiency of S there is a state, say α', in S such that

$$m_{\alpha'}(\mu_A([\lambda_1, \lambda)) \wedge \mu_B(J)) = 1$$

In this state α' we have

$$(*') \quad \operatorname{Var}(A, \alpha') \operatorname{Var}(B, \alpha') \leqslant h(\lambda_1, \lambda) h(\eta_1, \eta_2)$$

Repeat the "splitting of I" until the right-hand side of $(*)$ is less than the given $\varepsilon > 0$. This completes the proof. $\square$

We note that the converse of the above lemma does not hold in arbitrary quantum logics. Actually in the next Section we shall give an example of a quantum logic (L, S), L an orthomodular poset, S a full set of states on L, in which we have a pair of noncompatible observables C and D for which $\mathrm{Var}(C, \alpha)\mathrm{Var}(D, \alpha)=0$ for all α in S. Of course, the above Hilbert space description of a particle in a box provides such an example, too.

6.4. The Main Theorem. The axiom of Heisenberg claims the existence of such pairs of observables in O for which the condition (6.8) above holds. The following theorem characterizes observables of this kind. The proof of the theorem is contained in the foregoing discussion.

> *Theorem 6.1.* Let A and B be two observables in O satisfying the axiom of Heisenberg. Then, assuming the sufficiency of S, A and B are noncompatible; A and B are unbounded.

In the classical mechanical description of a physical system it is assumed that all the observables (dynamical variables) are compatible. Thus we have the following corollary.

> *Corollary.* The axiom of Heisenberg excludes the classical mechanical description of a physical system.

7. THE AXIOM OF COMPLEMENTARITY

In section 3 of this work we discussed Bohr's notion of complementarity, gave a definition of complementary physical quantities, and accepted the view that the existence of complementary physical quantities is an essential feature of the quantum mechanical description. In Section 4 we showed that in the Hilbert space formulation of quantum mechanics the complementarity of the canonically conjugate position and momentum operators is automatically guaranteed through the relation (4.2) of that section. In this section we shall discuss complementary physical quantities and the complementarity principle in a more general axiomatic framework for quantum mechanics.

7.1. Formulation of the Principle. Let L denote the logic of our physical system, i.e., the logic of the probability function p associated with F. We consider the map

$$\varphi: L\times L\rightarrow 2^{L}, (a, b)\rightarrow \varphi(a, b)=\{c\in L: c\leqslant a, c\leqslant b\} \tag{7.1}$$

which assigns to each pair of propositions the set of their lower bounds. We note that for any a and b in L:

If $\varphi(a, b)=\{0\}$, then $a\wedge b$ exists and $a\wedge b=0$.

If $\varphi(a, b)=(c]=\{x \in L: x \leqslant c\}$ for some c in L, then $a \wedge b$ exists and $a \wedge b=c$.
If a and b are compatible, with resolution $a=a_1 \vee c$, $b=b_1 \vee c$, then $a \wedge b$ exists and is equal to c. In this case $\varphi(a, b)=(c]$.
$\varphi(a, a)=\varphi(a, 1)=\varphi(1, a)=(a]$.
$\varphi(0, a)=\varphi(a, 0)=\{0\}$, $\varphi(1, 1)=L$.

Let $c \in \varphi(a, b)$, $c \neq 0$, and let α be a state in S such that $m_\alpha(c)=1$.[11] In this state α we know *a priori*—i.e., we do not have to make any measurements in order to assure this—that our system F has also the properties a and b. In other words, in this state α the experimental propositions $a=|A, E|$ and $b=|B, F|$ are true, i.e., their verification would lead with certainty to the "yes" result.

In accordance with the definition of complementary physical quantities, which we gave in section 3, we say that two experimental propositions a and b are complementary if no experimental arrangement exists by which they can simultaneously be verified. In other words, a and b are complementary, if it is not possible to know that the system F has simultaneously the properties a and b. Thus for any pair of complementary propositions a and b we must require: $\varphi(a, b)=\{0\}$, i.e., $a \wedge b$ exists and is equal to 0.

The lattice structure for L demands that $a \wedge b$ exists for any pair of propositions a and b in L. What is the physical reason for claiming that in every set $\varphi(a, b)$ of lower bounds of a and b there exists the greatest element, i.e., g.l.b.$(a, b)=a \wedge b$ is in $\varphi(a, b)$ for every a and b in L?

So far as we know there exist no physically ground motivations for the lattice assumption of the logic L. Evidently, the best known attempt in this direction are the filters constructed by J. M. Jauch (1968). A filter, which is intended to determine the meet of any two propositions, is composed of an infinite series of two alternating experiments. These filters have their origin in the well-known mathematical result of von Neumann (1950) which gives an algebraic construction for the meet of any two orthogonal projections E and F in a Hilbert space H (cf. Section 4).

Though we cannot give a physical interpretation for the lattice assumption of L, and hence we omit it, we must of course require that for any two complementary experimental propositions the set of their lower bounds does not contain any nonabsurd propositions.[12] However, using the notion of complementary propositions we can give the following

[11] Here, again, we assume the sufficiency of S.

[12] There are some other approaches to axiomatic quantum mechanics in which the lattice assumption does not appear so problematic, see, e.g., D. Finkelstein (1978) or W. Guz (1978).

characterization of the lattice assumption of L. For any two propositions a and b in L:

If a and b are complementary, then require $\varphi(a, b)=\{0\}$, i.e., $a \wedge b$ exists and is equal to 0.
If a and b are not complementary, but (i) they are compatible, then deduce (after definition) $a=a_1 \vee c$, $b=b_1 \vee c$, $\varphi(a, b)=(c]$, and $a \wedge b=c$, or (ii) they are noncompatible, but what about $\varphi(a, b)$?

Thus the problem of giving a physical interpretation for the lattice meet of any two propositions a and b concerns only noncompatible propositions which are not complementary, because for complementary propositions we must *a priori* assume it, and for compatible propositions we can *a posteriori* deduce it.

The above characterization of the lattice assumption also shows that the often-stated argument against the lattice structure is untenable. The argument goes as follows: Because of the uncertainty relation $\Delta x \Delta p \geqslant h$, the meet of the propositions $a=$"the position observable has a value in $[x_0, x_0+\Delta x]$" and $b=$"the momentum observable has a value in $[p_0, p_0+\Delta p]$" is undefinable whenever $\Delta x \Delta p \ll h$, and thus the lattice assumption cannot be made (see, e.g., Gudder, 1967). However, as we now see the problem, the propositions a and b are complementary for any Δx and Δp, and thus we require $a \wedge b=0$.

The existence of orthoposets which are not lattices and which do admit full sets of states on them makes the lattice question very relevant. Such an example is provided with the system (J_{18}, S) which is defined in Figure 1 and in Table 1. J_{18} is the first known example of an orthoposet which is not a lattice (Janowitz, 1963; see also Greechie, 1969).

In J_{18} the propositions c and d are noncompatible, but they are not complementary. *A fortiori*, the observables C and D, defined with the Boolean subalgebras $\{0, c, c^{\perp}, 1\}$ and $\{0, d, d^{\perp}, 1\}$ of J_{18} together with spectra $\sigma(C)=\{0,1\}$ and $\sigma(D)=\{0,1\}$ in R, are noncompatible and noncomplementary (see the definition below). We note also that if we equip J_{18} with the full set of states S defined in table 1 we have Var (C, α) Var $(D, \alpha)=0$ for all α in S.

In accordance with the concept of complementary experimental propositions we say that two observables A and B in O are complementary if for any bounded Borel sets E and F in $B(R)$ such that $E \cap \sigma(A) \subsetneqq \sigma(A)$ and $F \cap \sigma(B) \subsetneqq \sigma(B)$ we have $\varphi(\mu_A(E), \mu_B(F))=\{0\}$, i.e., $\mu_A(E) \wedge \mu_B(F)$ exists and is equal to 0. Note that the assumption "$E \cap \sigma(A) \subsetneqq \sigma(A)$ and $F \cap \sigma(B) \subsetneqq \sigma(B)$" is needed in the above definition to avoid the conclusion that our complementary observables are unbounded.

In Section 3 we adopted the view that the existence of complementary

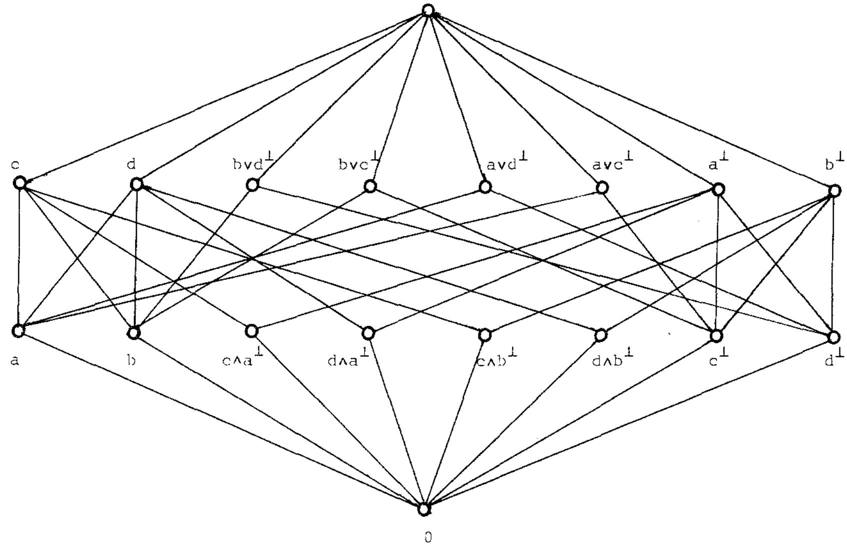

Fig. 1. J_{18}, an orthomodular poset.

physical quantities is an essential feature of the quantum mechanical description. This attitude we expressed in the complementarity principle. Following these ideas we adopt the following axiom in which we demand the existence of complementary pairs of observables in O.

The Axiom of Complementarity. There exists in O at least a pair of nonconstant complementary observables.

7.2. The Main Results

Theorem 7.1. Let A and B be two complementary observables in O. A and B are compatible only when either A or B is a constant observable.

Proof. [13] Let A be a constant observable, i.e., an observable with the range $\{0,1\}$ in L. Clearly A and B are complementary and compatible. On the other hand, let A and B be complementary and compatible. Because A and B are compatible there is an observable C and Borel functions f and g such that $A=f(C)$ and $B=g(C)$. In particular, this means that the Boolean sub σ algebras $\{\mu_A(E)\colon E\in B(R)\}$ and $\{\mu_B(F)\colon F\in B(R)\}$ of L are contained in the Boolean sub σ algebra $\{\mu_C(E)\colon E\in B(R)\}$ of L. Suppose that the spectra of A and B are unbounded. In this case we may write

[13] The author is indebted to Professor E. Beltrametti for pointing out a deficiency in the original proof of the theorem.

TABLE 1. A full set of states $S=\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_7,\sigma_8\}$ on J_{18}.

	a	$a^\perp$	b	$b^\perp$	c	$c^\perp$	d	$d^\perp$	$c\wedge a^\perp$	$c^\perp\vee a$	$d\wedge a^\perp$	$d^\perp\vee a$	$c\wedge b^\perp$	$c^\perp\vee b$	$d\wedge b^\perp$	$d^\perp\vee b$
σ_1	1	0	0	1	1	0	1	0	0	1	0	1	1	0	1	0
σ_2	0	1	1	0	1	0	1	0	1	0	1	0	0	1	0	1
σ_3	0	1	0	1	0	1	1	0	0	1	1	0	0	1	1	0
σ_4	0	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1
σ_5	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
σ_6	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
σ_7	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0
σ_8	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0

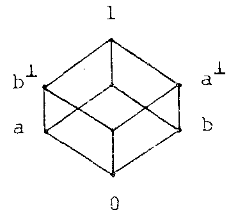

Fig. 2.

$$
\begin{aligned}
1 &= 1 \wedge 1 = \mu_A(\sigma(A)) \wedge \mu_B(\sigma(B)) = \mu_A(R) \wedge \mu_B(R) \\
&= \mu_A\left(\bigcup_{k=-\infty}^{+\infty} (k, k+1]\right) \wedge \mu_B\left(\bigcup_{1=-\infty}^{+\infty} (1, 1+1]\right) \\
&= (\vee_k \mu_A((k, k+1])) \wedge (\vee_1 \mu_B((1, 1+1])) \\
&= \vee_k(\vee_1(\mu_A((k, k+1]) \wedge \mu_B((1, 1+1]))) = \vee_k(\vee_1 0) = 0
\end{aligned}
$$

which is a contradiction. The case of bounded spectrum (spectra) reduces immediately to the above case. This completes the proof. □

The above theorem gives the existence of noncompatible observables in O. Hence we have again a corollary.

> *Corollary.* The axiom of complementarity excludes the classical mechanical description of a physical system.

We consider next a simple example, which further clarifies the content of the axiom of complementarity. So let us consider two elementary observables A and B in O with ranges $\{0, a, a^\perp, 1\}$ and $\{0, b, b^\perp, 1\}$ in L, and with spectra $\sigma(A) = \{1,2\}$ and $\sigma(B) = \{1,2\}$ in R, respectively. If $a \leqslant b^\perp$, i.e., if a and b are orthogonal, and thus $a \wedge b = 0$, then the observables A and B are compatible. The Boolean subalgebra of L generated by the union of the ranges of A and B being $\{0, a, a^\perp, b, b^\perp, a \vee b, a^\perp \wedge b^\perp, 1\}$ (see Figure 2). Because $a \wedge b^\perp = a \neq 0$, i.e., $\mu_A(\{1\}) \wedge \mu_B(\{2\}) = a \wedge b^\perp = a \neq 0$, we find that the observables A and B are noncomplementary, which is in conformity with Theorem 7.1 above.

Thus we conclude that the axiom of complementarity implies, in fact, the existence of at least two propositions, say a and b, in L, such that $a \wedge b = 0$, but a and b are not orthogonal.

Let A and B be a pair of nonconstant complementary observables. With any two Borel functions f and g of the real line R we define another pair of observables $C = f(A)$ and $D = g(B)$ with the property

$$\mu_C(E)\wedge\mu_D(F)=\mu_A(f^{-1}(E))\wedge\mu_B(g^{-1}(F)), \qquad \forall E, F\in B(R)$$

Because

$$E\subset\sigma(C)\Rightarrow f^{-1}(E)\subset\sigma(A) \text{ and } F\subset\sigma(D)\Rightarrow g^{-1}(F)\subset\sigma(B)$$

we find that the observables C and D are also complementary provided the functions f and g are regular enough (i.e., the inverse images of any bounded Borel set under f and g are bounded). We do not lack such functions. Thus we conclude that any pair of nonconstant complementary observables generates a family of such pairs.

We close this section with a theorem which we use in studying the "logical status" of the axiom of complementarity. In this theorem we assume the lattice structure for L. This result is closely connected with the classic axiomatization of Boolean algebras given by E. V. Huntington according to which a Boolean algebra is a complemented lattice in which the complementation is a pseudocomplementation (see e.g., Grätzer, 1978).

Theorem 7.2. An ortholattice L is Boolean if and only if the following condition is satisfied:

$$a\wedge b=0\Rightarrow a\perp b, \qquad \forall a, b\in L \tag{7.2}$$

Proof. Suppose that L is Boolean, but there exist a and b in L such that $a\wedge b=0$, $a\not\perp b$. Because $a=a\wedge 1=a\wedge(b\vee b^{\perp})=(a\wedge b)\vee(a\wedge b^{\perp})=a\wedge b^{\perp}$, we have $a\leqslant b^{\perp}$, i.e. $a\perp b$. A contradiction. So (7.2) is necessary. It is also sufficient, because for any a and b in L we have

$$a=(a\wedge b)\vee((a\wedge b)^{\perp}\wedge a) \quad \text{and} \quad b=(a\wedge b)\vee((a\wedge b)^{\perp}\wedge b)$$

which, assuming (7.2), shows that a and b are compatible. □

The axiom of complementarity—which is physically based on the nonzeroness of h, and which states the existence of complementary physical quantities—implies thus a non-Boolean structure for L. This result is very useful in discussing the status of complementarity in quantal description as well as in comparing the uncertainty principle and the complementarity principle in our formulations.

8. THE MUTUAL INDEPENDENCE OF THE AXIOMS

In this section we shall show that the axiom of Heisenberg and the axiom of complementarity are logically independent axioms. Thus we have

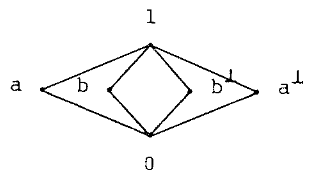

Fig. 3.

clear evidence that the two central principles of quantum theory, the uncertainty principle and the complementarity principle, are independent principles. This result is especially important because in reading the original papers of Bohr and Heisenberg, as well as some more recent writings, one easily gets confused about the interrelation of these two principles.

As we have learned in Section 3, Bohr interpreted uncertainty relations "as a simple symbolical expression for the complementary nature of the space–time description and the claim of causality" (Bohr 1927/1978, p. 60). The view that the uncertainty relations exhibit a mathematical expression for complementarity is quite generally accepted (see Section 3). For example, A. Messiah teaches us that Heisenberg's uncertainty relations are a very general consequence of the statistical interpretation of the wave–particle duality (Messiah, 1961, p. 116).

Though Heisenberg wrote in his reminiscence (Heisenberg, 1967) that "the uncertainty relations were just a special case of the more general complementarity principle," it is, however, evident from his earlier writings (see especially Heisenberg 1927, 1949, 1955) that he did not regard wave–particle duality, the very origin of Bohr's notion of complementarity, as a presupposition for uncertainty relations. A view conflicting with the one mentioned above can be read in Bernard d'Espagnat's recent book where he writes that "the Heisenberg uncertainty relations can be considered as a generalization of the complementarity principle" (d'Espagnat, 1976, p. 253).

Let us consider two simple observables A and B in O defined by

$$\text{range}\,(A)=\{0, a, a^{\perp}, 1\}, \qquad \sigma(A)=\{\lambda_1, \lambda_2\}$$

$$\text{range}\,(B)=\{0, b, b^{\perp}, 1\}, \qquad \sigma(B)=\{\lambda_1', \lambda_2'\}$$

See Figure 3 for definitions of A and B. We note that A and B correspond to the photon polarization experiments with polarization angles, say, ϕ and ϕ' such that $\phi \neq \phi'$ (mod π) (see, e.g., Piron, 1976). Clearly A and B are complementary observables. Note also that they are noncompatible (cf. the example in Section 7). Let $\alpha \in S$ be an eigenstate of A corresponding to the eigenvalue λ_1. In this state α we have $\mathrm{Var}(A, \alpha) = 0$ and hence $\mathrm{Var}(A, \alpha)\,\mathrm{Var}(B, \alpha) = 0$. Thus we have the following theorem.

Theorem 8.1. The axiom of complementarity does not imply the axiom of Heisenberg.

Next we shall discuss the matter from the reverse viewpoint. Suppose that A and B are two observables in O satisfying the axiom of Heisenberg. If there exist bounded Borel sets E and F in $B(R)$ such that $\varphi(\mu_A(E), \mu_B(F)) \neq \{0\}$, then, assuming the sufficiency of S, there exists a state α in $S_A^V \cap S_B^V$ such that $m_\alpha \circ \mu_A(I) = 1$ and $m_\alpha \circ \mu_B(J) = 1$, where $I = [-\lambda, \lambda]$ is the smallest closed interval containing E and $J = [-\lambda', \lambda']$ is the smallest closed interval containing F. In this state α we have $\mathrm{Var}(A, \alpha) \leqslant \lambda^2$ and $\mathrm{Var}(B, \alpha) \leqslant \lambda'^2$. On the other hand, for every state α in $S_A^V \cap S_B^V$ we also have $\mathrm{Var}(A, \alpha)\,\mathrm{Var}(B, \alpha) \geqslant h^2$ for a given positive real number h. This means that $(\lambda\lambda')^2 \geqslant h^2$. This shows that the axiom of Heisenberg does not exclude the possibility of A and B not being complementary. However, the axiom of Heisenberg puts a limitation on the "size" of the sets E and F for which we may have $\varphi(\mu_A(E), \mu_B(F)) \neq \{0\}$. This limitation is $(\lambda\lambda')^2 \geqslant h^2$, where λ and λ' are defined as above.

An example of observables of this kind was given by S. Bugajski (1978). In the following we shall reproduce this example in a slightly modified form, which is more suitable for our purposes.

We consider a particle moving in one dimension. The classical Hamiltonian description of this system is carried out in the phase space $M = R^2$, whose Borel structure $B(M) = B(R^2)$ describes in a natural way the propositional system of the particle. We denote it as L. The dynamic variables of the particle are expressed as real valued Borel measurable functions on M, i.e., as mappings $f: M \to R$ with property $f^{-1}(E) \in B(M)$ whenever $E \in B(R)$. For example, the position coordinate and the momentum coordinate of the particle are defined as $f_q: M \to R, (q, p) \to f_q(q, p) = q$ and $f_p: M \to R, (q, p) \to f_p(q, p) = p$, respectively. The σ homomorphisms induced by the dynamic variables of the particle are called the observables of the particle. We denote the set of all observables $\{A : B(R) \to L, A\ \sigma \text{ homomorphism}\}$ as O. For example, the position observable and the momentum observable of the particle are defined as $Q: B(R) \to L, E \to Q(E) = f_q^{-1}(E)$ and $P: B(R) \to L, E \to P(E) = f_p^{-1}(E)$, respectively. A state of the particle is defined as a point of M. The points of M can be identified with unit measures on L. To allow also nontrivial probability measures on L as states of the particle (mixed states) we generalize thus: a state of the particle is a probability measure on L. We denote the set of all states

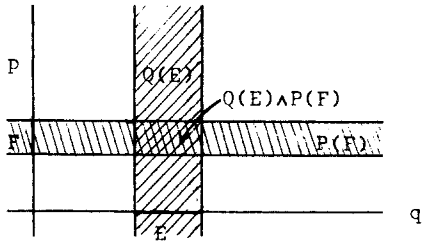

Fig. 4.

$\{\alpha: L \to [0,1],\ \alpha$ probability measure$\}$ as S. As usual the function $p: O \times S \times B(R) \to [0,1]$ is defined through the relation $p(A, \alpha, E) = \alpha(A(E))$ for every A in O, α in S, and E in $B(R)$. Thus the system (M, L, O, S, p) specifies the classical mechanical description of our physical system.

Next we fix a nonnegative real number h, and define a subset S_h of S as

$$S_h = \{\alpha \in S : \mathrm{Var}(Q, \alpha)\mathrm{Var}(P, \alpha) \geqslant h^2\}$$

That the set S_h is order determining, i.e., full, and convex is shown by Bugajski (1978). Thus we have a theory (M, L, O, S_h, p) in which

$$\mathrm{Var}(Q, \alpha)\mathrm{Var}(P, \alpha) \geqslant h^2 \quad \text{for all } \alpha \text{ in } S_h \tag{8.1}$$

but in which for every nonvoid E and F in $B(R)$

$$Q(E) \wedge P(F) = E \times F \neq \varnothing \tag{8.2}$$

(see Figure 4.)

The above example suggests a very interesting viewpoint of the nature of the quantal description, which we shall develop in the following chapter. We conclude the above discussion with the theorem, which closes the question of the mutual independence of our axioms.

> *Theorem 8.2.* The axiom of Heisenberg does not imply the axiom of complementarity.

9. DISCUSSION

The uncertainty principle and the complementarity principle embrace characteristic features of the quantum theory. These two principles, as mathematicized in this work, are logically independent. In this final section we shall discuss some features of these principles and develop a view of the nature of quantal description which arises out of the preceding discussion. Moreover, we shall show that our formulations of these principles are logically independent of the Jauch formulation of the superposition principle.

The complementarity principle is physically based on the nonzeroness of h, and it claims the existence of complementary physical quantities. This principle was shown to imply a non-Boolean structure for the proposition system L. In fact, as the proof of the Theorem 7.2 in Section 7.2 indicates, the axiom of complementarity leads to the break of the distributive law. Thus we can say that the quantal feature of the description resulting from the complementarity principle is incorporated in the non-Boolean proposition structure L.

We next discuss the role of the uncertainty principle in the description of any physical system. We begin with the theory (M, L, O, S, p) which specifies the classical mechanical description of the system concerned (cf. p. 832).

We now transfer to the quantum mechanical description, i.e., we suppose that the actions involved in our problem are comparable with the quantum of action h, and hence h is now relevant in our description. In this case "it is," according to Heisenberg, "impossible to obtain an exact determination of the simultaneous values of two variables, but rather that there is a lower limit to the accuracy with which they can be known" (Heisenberg, 1930/1949, p. 3).

We proceed by postulating "this lower limit to the accuracy with which certain variables can be known simultaneously" as a law of nature. This means that from the above set of states S, which describes all the classically possible experimental arrangements, we must pick out the states, i.e., experimental arrangements, which fulfill the axiom of Heisenberg. We do it by choosing $S_h=\{\alpha\in S:\operatorname{Var}(Q,\alpha)\operatorname{Var}(P,\alpha)\geqslant h^2\}$. So we still have the phase space M, the Boolean proposition structure $L=B(M)$, and the dynamic variables expressed as measurable functions $M\to R$. But now the set of physical states S_h is essentially different from the original S. S_h is order determining and convex, but it does not contain any pure states. Restricting S to S_h the position and the momentum observables Q and P automatically satisfy the uncertainty relation

$$\operatorname{Var}(Q,\alpha)\operatorname{Var}(P,\alpha)\geqslant h^2 \quad \text{for all } \alpha \text{ in } S_h$$

but they are not complementary, because $Q(I)\wedge P(J)\neq\varnothing$ for any nonvoid intervals I and J of R. So the quantal feature of the above description is entirely incorporated in the state system S_h, and the effect of the uncertainty principle is only to deform the concept of state, i.e., to restrict S to S_h.

We are now in a position to advocate the following view of the nature of quantal description.

Schematically:

	→ uncertainty	→ generalization of the concept of state	→ "uncertainty description"
quantum of action h			
	→ complementarity →	degeneration of the Boolean proposition structure	→ "complementary description"

In words. Quantum theory is the theory of quanta. The finite magnitude of the quantum of action h is the physical reason for the uncertainty principle as well as for the complementarity principle. The uncertainty principle and the complementarity principle, as formulated in this work, are logically independent principles. On the one hand, the uncertainty principle leads to a natural generalization of the classical concept of state that is meaningful in quantum mechanics, too. On the other hand, the complementarity principle implies that the Boolean σ-algebra structure of the set of all propositions in classical mechanics degenerates into a non-Boolean structure.

The effect of the uncertainty principle is thus to deform the concept of state, whereas the complementarity principle leads to a nondistributive propositional structure. (A somewhat similar view was developed in Lahti 1976a.) This is satisfactory because the uncertainty principle is of a statistical character, and the complementarity principle is of a nonstatistical character.

We are now also in a position to distinguish between two kinds of description, namely, the "uncertainty description" and the "complementary description," both of which embrace an important feature of the quantal description, but which only together provide the proper quantum mechanical description of any physical system.

Throughout this work we have given some examples of quantum logics which do satisfy either the axiom of Heisenberg or the axiom of complementarity but not both. If we adopt the view that in the proper quantum mechanical description of any physical system both of the two axioms are to be satisfied, we have to conclude that the quantum logics referred to do not provide examples of proper quantum mechanical descriptions.

We close this work with a comparison of the superposition principle, as formulated by Jauch (1968), with the complementarity principle and the uncertainty principle, as formulated in this work. We shall conclude the following:

(1) The complementarity principle does not imply
the uncertainty principle.
(2) The uncertainty principle does not imply
the complementarity principle.
(3) The complementarity principle does not imply
the superposition principle.
(4) The superposition principle does not imply
the complementarity principle.
(5) The uncertainty principle does not imply
the superposition principle.
(6) The superposition principle does not imply
the uncertainty principle.

The above stated interrelations of the three fundamental principles of quantum theory seem to be quite satisfactory.

We have already given some arguments in favor of the independence of the complementarity principle and the uncertainty principle. Equally, the independence of the uncertainty principle and the superposition principle appears to be acceptable.

The appearance of the superposition principle in quantum mechanics is said to result from the wave–particle duality, through the work of Louis de Broglie in 1924. Niels Bohr developed his notion of complementarity from this fundamental duality, too. The discovery of superselection rules (Wick, Wightman, and Wigner, 1952; see also Jauch 1968, and Beltrametti and Cassinelli, 1976) has revealed that the superposition principle is not a universally valid principle, whereas it is quite generally thought that the complementarity principle, as well as the uncertainty principle, is of that general nature. Thus the relation (3), and also (5), above appears to be acceptable, too. Though the relation (4) is, of course, acceptable as a mathematical fact, we have to admit that its physical content is not so clear to us. We now leave this question open.

In Dirac's formulation of quantum mechanics one assumes the following:

> "each state of a dynamical system at a particular time corresponds to a ket vector, the correspondence being such that if a

state results from the superposition of certain other states, its corresponding ket vector is expressible linearly in terms of the corresponding ket vectors of the other states, and converly" (Dirac, 1930/1958, p. 16).

Thus if the state R can be formed by superposition of states A and B, then the corresponding ket vectors $|R\rangle$, $|A\rangle$, and $|B\rangle$ are connected by

$$|R\rangle = c_1|A\rangle + c_2|B\rangle \tag{9.1}$$

for some complex numbers c_1 and c_2. The equation (9.1) together with the above citation of Dirac indicates that "the superposition relationship is symmetrical between all the three states A, B and R" (Dirac 1930/1958, p. 16).

Let L be the proposition system of a given physical system. In order to give Jauch's formulation for the superposition principle one has to assume that L is an orthomodular lattice containing atoms. According to Jauch (1968) the proposition system L satisfies the superposition principle if for every pair of distinct atoms e_1 and e_2 in L there is a third atom e_3 in L, distinct from e_1 and e_2, such that

$$e_1 \vee e_2 = e_1 \vee e_3 = e_2 \vee e_3 \tag{9.2}$$

The similarity of the above two formulations for the superposition principle is remarkable. In the following we shall discuss only the Jauch formulation of the superposition principle without explicitly mentioning it every time.

An immediate consequence of the superposition principle is the break of the distributive law in L. Really, assuming the distributive law to hold in L one would conclude that $e_1 = e_1 \wedge (e_1 \vee e_2) = e_1 \wedge (e_3 \vee e_2) = (e_1 \wedge e_3) \vee (e_1 \wedge e_2) = 0 \vee 0 = 0$, which is a contradiction. Thus both the complementarity principle and the superposition principle imply a nondistributive structure for L.

Previously we argued that the complementarity principle, or more precisely the axiom of complementarity, implies the following:

Property 9.1. There exist in L, at least, two propositions, say, a and b such that $a \wedge b = 0$, but $a \not\perp b$

Let e_1, e_2, and e_3 be three distinct atoms of L satisfying (9.2). So we have $e_1 \wedge e_2 = e_1 \wedge e_3 = e_2 \wedge e_3 = 0$, and $e_1 \vee e_2 = e_1 \vee e_3 = e_2 \vee e_3$. Assume that $e_1 \perp e_2$ and $e_1 \perp e_3$, which implies that e_1 is compatible both with e_2 and e_3. In this case we would have $e_1 = e_1 \wedge (e_1 \vee e_3) = e_1 \wedge (e_2 \vee e_3) = (e_1 \wedge e_2) \vee (e_1$

$\wedge e_3) = 0$, which is a contradiction. So we conclude that the superposition principle, too, implies Property 9.1 for the proposition system L.

We conclude that a necessary condition that a given proposition system L satisfies either the complementarity principle or the superposition principle is that L possesses Property 9.1. Property 9.1, which the complementarity principle and the superposition principle share, may well be connected to the common root, the fundamental wave–particle duality, of these two principles.

The well-known orthomodular lattice D_{16} (see, e.g., Greechie and Gudder, 1973) provides us with an example of proposition systems which do not satisfy the superposition principle, but in which we have complementary physical quantities. Conversely, every finite-dimensional Hilbert space, e.g., the three-dimensional euclidean space R^3, provides us with an example of quantum logics satisfying the superposition principle, but which do not admit complementary physical quantities.

The examples which we gave to show the independence of the complementarity principle and the uncertainty principle will also indicate the independence of the superposition principle and the uncertainty principle.

We conclude that the above relations (1)–(6) are, at least, well-established mathematical facts.

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